Pressure Vessels

Cylindrical or spherical pressure vessels (e.g., hydraulic cylinders, gun barrels, pipes, boilers and tanks) are commonly used in industry to carry both liquid s and gases under pressure. When the pressure vessel is exposed to this pressure, the material comprising the vessel is subjected to pressure loading, and hence stresses, from all directions. The normal stresses resulting from this pressure are functions of the radius of the element under consideration, the shape of the pressure vessel (i.e., open ended cylinder, closed end cylinder, or sphere) as well as the applied pressure.

Two types of analysis are commonly applied to pressure vessels. The most common method is based on a simple mechanics approach and is applicable to "thin wall" pressure vessels which by definition have a ratio of inner radius, r, to wall thickness, t, of r/t≥10. The second method is based on elasticity solution and is always applicable regardless of the r/t ratio and can be referred to as the solution for "thick wall" pressure vessels. Both types of analysis are discussed here, although for most engineering applications, the thin wall pressure vessel can be used.

Thin-Walled Pressure Vessels

Several assumptions are made in this method.

1) Plane sections remain plane
2) r/t ≥ 10 with t being uniform and constant
3) The applied pressure, p, is the gage pressure (note that p is the difference between the absolute pressure and the atmospheric pressure)
4) Material is linear-elastic, isotropic and homogeneous.
5) Stress distributions throughout the wall thickness will not vary
6) Element of interest is remote from the end of the cylinder and other geometric discontinuities.
7) Working fluid has negligible weight

Cylindrical Vessels: A cylindrical pressure with wall thickness, t, and inner radius, r, is considered, (see Figure 1). A gauge pressure , p, exists within the vessel by the working fluid (gas or liquid). For an element sufficiently removed from the ends of the cylinder and oriented as shown in Figure 1, two types of normal stresses are generated: hoop, σ_h, and axial, σ_a, that both exhibit tension of the material.

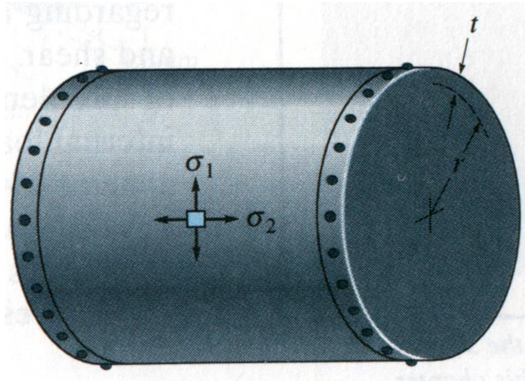


Figure 1 Cylindrical Thin-Walled Pressure Vessel

For the hoop stress, consider the pressure vessel section by planes sectioned by planes a, b, and c for Figure 2. A free body diagram of a half segment along with the pressurized working fluid is shown in Fig. 3 Note that only the loading in the x-direction is shown and that the internal reactions in the material are due to hoop stress acting on incremental areas, A, produced by the pressure acting on projected area, A_p. For equilibrium in the x-direction we sum forces on the incremental segment of width dy to be equal to zero such that:

$$\sum F_x = 0$$

$$2[\sigma_h A] - pA_p = 0 = 2[\sigma_h \text{ t dy}] - \text{p 2r dy} \tag{1}$$

or solving for σ_h

$$\sigma_h = \frac{pr}{t}$$

where dy = incremental length, t = wall thickness, r = inner radius, p = gauge pressure, and σ_h is the hoop stress.

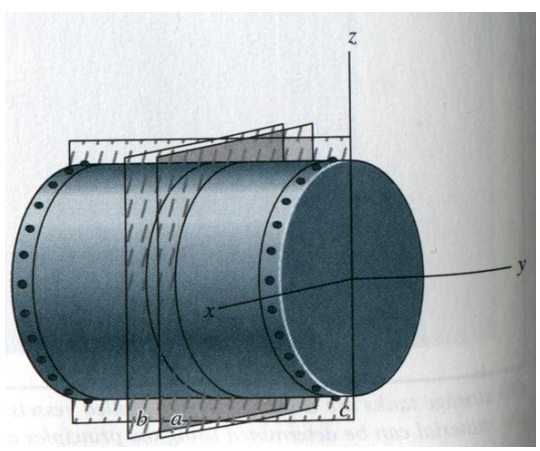


Figure 2 Cylindrical Thin-Walled Pressure Vessel Showing Coordinate Axes and Cutting Planes (a, b, and c)

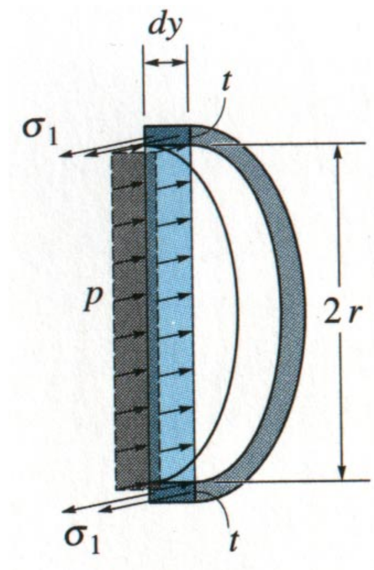


Figure 3 Free-Body Diagram of Segment of Cylindrical Thin-Walled Pressure Vessel Showing Pressure and Internal Hoop Stresses

For the axial stress, consider the left portion of section b of the cylindrical pressure vessel shown in Figure 2. A free body diagram of a half segment along with the pressurized working fluid is shown in Fig. 4 Note that the axial stress acts uniformly throughout the wall and the pressure acts on the endcap of the cylinder. For equilibrium in the y-direction we sum forces such that:

$$\sum F_y = 0$$

$$\sigma_a A - pA_e = 0 = \sigma_a \pi\left(r_o^2 - r^2\right) - \mathrm{p}\ \pi r^2$$

or solving for σ_a

$$\sigma_a = \frac{\mathrm{p}\ \pi r^2}{\pi\left(r_o^2 - r^2\right)}$$

substituting $\mathrm{r_o} = \mathrm{r} + \mathrm{t}$ gives (2)

$$\sigma_a = \frac{\mathrm{p}\ \pi r^2}{\pi\left([\mathrm{r}+\mathrm{t}]^2 - r^2\right)} = \frac{\mathrm{p}\ \pi r^2}{\pi\left(r^2 + 2rt + t^2 - r^2\right)} = \frac{\mathrm{p}\ r^2}{\left(2rt + t^2\right)}$$

since this is a thin wall with a small t,t^2 is smaller and can be neglected such that after simplification

$$\sigma_a = \frac{\mathrm{p}\ r}{2t}$$

where $\mathrm{r_o}$ = inner radius and σ_a is the axial stress.

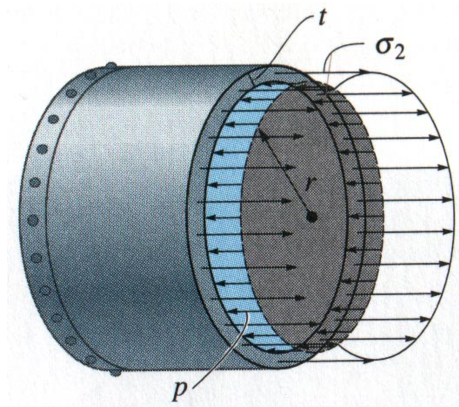


Figure 4 Free-Body Diagram of End Section of Cylindrical Thin-Walled Pressure Vessel Showing Pressure and Internal Axial Stresses

Note that in Equations 1 and 2, the hoop stress is twice as large as the axial stress. Consequently, when fabricating cylindrical pressure vessels from rolled-formed plates, the longitudinal joints must be designed to carry twice as much stress as the circumferential joints.

Spherical Vessels: A spherical pressure vessel can be analyzed in a similar manner as for the cylindrical pressure vessel. As shown in Figure 5, the "axial" stress results from the action of the pressure acting on the projected area of the sphere such that

$$\sum F_y = 0$$

$$\sigma_a A - pA_e = 0 = \sigma_a \pi\left(r_o^2 - r^2\right) - \mathrm{p}\ \pi r^2$$

or solving for σ_a

$$\sigma_a = \frac{\mathrm{p}\ \pi r^2}{\pi\left(r_o^2 - r^2\right)}$$

substituting $\mathrm{r_o} = \mathrm{r} + \mathrm{t}$ gives (3)

$$\sigma_a = \frac{\mathrm{p}\ \pi r^2}{\pi\left([\mathrm{r}+\mathrm{t}]^2 - r^2\right)} = \frac{\mathrm{p}\ \pi r^2}{\pi\left(r^2 + 2rt + t^2 - r^2\right)} = \frac{\mathrm{p}\ r^2}{\left(2rt + t^2\right)}$$

since this is a thin wall with a small t,t^2 is smaller and can be neglected such that after simplification

$$\sigma_a = \frac{\mathrm{p}\ r}{2t} = \sigma_h$$

Note that for the spherical pressure vessel, the hoop and axial stresses are equal and are one half of the hoop stress in the cylindrical pressure vessel. This makes the spherical pressure vessel a more "efficient" pressure vessel geometry.

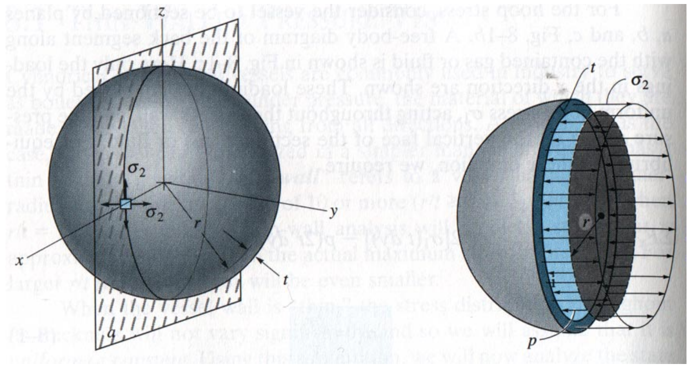


Figure 5 Free-Body Diagram of End Section of Spherical Thin-Walled Pressure Vessel Showing Pressure and Internal Hoop and Axial Stresses

The analyses of Equations 1 to 3 indicate that an element in either a cylindrical or a spherical pressure vessel is subjected to biaxial stress (i.e., a normal stress existing in only two directions). In reality, the element is subjected to a radial stress, σ_r which acts along a radial line. The stress has a compressive value equal to the pressure, p, at the inner wall, and decreases through the wall to zero at the outer wall (plane stress condition) since the gage pressure there is zero. For thin walled pressure vessels, the radial component is assumed to equal zero throughout the wall since the limiting assumption of r/t=10 results in σ_h being 10 times greater than σ_r=p and σ_a being 5 time greater than σ_r=p. Note also that the three normal stresses are principal stresses and can be used directly to determine failure criteria.

Note that the relations of Equation 1 to 3 are for internal gauge pressures only. If the pressure vessel is subjected to an external pressure, it may cause the pressure vessel to become unstable and collapse may occur by buckling of the wall.

Thick-Walled Pressure Vessels

Closed-form, analytical solutions of stress states can be derived using methods developed in a special branch of engineering mechanics called elasticity. Elasticity methods are beyond the scope of the course although elasticity solutions are mathematically exact for the specified boundary conditions are particular problems. For cylindrical pressure vessels subjected to an internal gage pressure only the following relations result:

$$\sigma_h = \frac{r_i^2 p}{\left(r_o^2 - r_i^2\right)}\left(1 + \frac{r_o^2}{r^2}\right)$$

$$\sigma_a = \frac{r_i^2 p}{\left(r_o^2 - r_i^2\right)} \qquad (4)$$

$$\sigma_r = \frac{r_i^2 p}{\left(r_o^2 - r_i^2\right)}\left(1 - \frac{r_o^2}{r^2}\right)$$

where r_o=outer radius, r_i=inner radius, and r is the radial variable. Equations 4 apply for any wall thickness and are not restricted to a particular r/t ratio as are the Equations 1 and 2. Note that the hoop and radial stresses(σ_h and σ_r) are functions of r (i.e. vary through the wall thickness) and that the axial stress, σ_a, is independent of r (i.e., is constant through the wall thickness. Figure 6 shows the stress distributions through the wall thickness for the hoop and radial stresses. Note that for the radial stress distributions, the maximum and minimum values occur, respectively, at the outer wall (σ_r=0) and at the (σ_r=-p) as noted already for the thin walled pressure vessel.

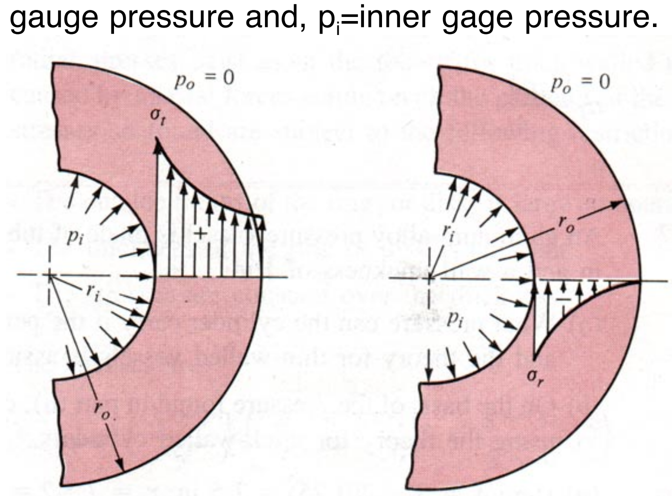
Equations 4 can be generalized for the case of internal and external pressures such that

$$\sigma_h = \frac{r_i^2 p_i - r_o^2 p_o - r_i^2 r_o^2 (p_o - p_i)/r^2}{\left(r_o^2 - r_i^2\right)}$$

$$\sigma_h = \frac{r_i^2 p_i - r_o^2 p_o}{\left(r_o^2 - r_i^2\right)} \qquad (5)$$

$$\sigma_h = \frac{r_i^2 p_i - r_o^2 p_o + r_i^2 r_o^2 (p_o - p_i)/r^2}{\left(r_o^2 - r_i^2\right)}$$

where p_o=is the outer gauge pressure and, p_i=inner gage pressure.



a) hoop stress b) radial stress

Figure 6 Stress distributions of hoop and radial stresses