Potential Field Methods

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Potential Field Methods

Basic Idea:

- robot is represented by a point in C-space
- treat robot like particle under the influence of an **artificial potential field U**
- **U** is constructed to reflect (locally) the structure of the free C-space (hence called 'local' methods)
- originally proposed by Khatib for on-line collision avoidance for a robot with proximity sensors

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Motion planning is an iterative process

1. compute the artificial force $\vec{F}(\mathbf{q}) = -\nabla \mathbf{U}(\mathbf{q})$ at current configuration
2. take a small step in the direction indicated by this force
3. repeat until reach goal configuration (or get stuck)

Note:

- major problem: local minima (most potential field methods are incomplete)
- advantages: speed
- RPP, a randomized potential field method proposed by Barraquand and Latombe for path planning, can be applied to robots with many dof

The Potential Field (translation only)

Assumption: $\mathcal{A}$ translates freely in $\mathcal{W} = \mathbb{R}^2$ or $\mathbb{R}^3$ at fixed orientation (so $\mathcal{C} = \mathcal{W}$)

The Potential Function: $\mathbf{U} : \mathcal{C}_{free} \longrightarrow \mathbb{R}^1$

- want robot to be *attacted* to goal and *repelled* from obstacles
  - **attractive potential** $\mathbf{U}_{att}(\mathbf{q})$ associated with $\mathbf{q}_{goal}$
  - **repulsive potential** $\mathbf{U}_{rep}(\mathbf{q})$ associated with $\mathcal{CB}$
  - $\mathbf{U}(\mathbf{q}) = \mathbf{U}_{att}(\mathbf{q}) + \mathbf{U}_{rep}(\mathbf{q})$
- $\mathbf{U}(\mathbf{q})$ must be differentiable for every $\mathbf{q} \in \mathcal{C}_{free}$

The Field of Artificial Forces: $\vec{F}(\mathbf{q}) = -\nabla\mathbf{U}(\mathbf{q})$

- $\nabla\mathbf{U}(\mathbf{q})$ denotes gradient of $\mathbf{U}$ at $\mathbf{q}$, i.e., $\nabla\mathbf{U}(\mathbf{q})$ is a vector that 'points' in the direction of 'fastest change' of $\mathbf{U}$ at configuration $\mathbf{q}$
- e.g., if $\mathcal{W} = \mathbb{R}^2$, then $\mathbf{q} = (x, y)$ and

$$\nabla\mathbf{U}(\mathbf{q}) = \begin{bmatrix} \frac{\partial\mathbf{U}}{\partial x} \\ \\ \frac{\partial\mathbf{U}}{\partial y} \end{bmatrix}$$

- $|\nabla\mathbf{U}(\mathbf{q})| = \sqrt{(\frac{\partial\mathbf{U}}{\partial x})^2 + (\frac{\partial\mathbf{U}}{\partial y})^2}$ is the magnitude of the rate of change
- $\vec{F}(\mathbf{q}) = -\nabla\mathbf{U}_{att}(\mathbf{q}) - \nabla\mathbf{U}_{rep}(\mathbf{q})$

The Attractive Potential

Basic Idea: $\mathbf{U}_{att}(\mathbf{q})$ should **increase** as $\mathbf{q}$ moves **away from** $\mathbf{q}_{goal}$ (like potential energy increases as you move away from earth's surface)

Naive Idea: $\mathbf{U}_{att}(\mathbf{q})$ is linear function of distance from $\mathbf{q}$ to $\mathbf{q}_{goal}$

- $\mathbf{U}_{att}(\mathbf{q})$ does increase as move away from $\mathbf{q}_{goal}$
- but $-\nabla\mathbf{U}_{att}$ has constant magnitude so it doesn't help us get to the goal

Better Idea: $\mathbf{U}_{att}(\mathbf{q})$ is a 'parabolic well'

- $\mathbf{U}_{att}(\mathbf{q}) = \frac{1}{2}\xi\rho^2_{goal}(\mathbf{q})$, where
  - $\rho_{goal}(\mathbf{q}) = \|\mathbf{q} - \mathbf{q}_{goal}\|$, i.e., Euclidean distance
  - ξ is some positive constant scaling factor
- unique minimum at $\mathbf{q}_{goal}$, i.e., $\mathbf{U}_{att}(\mathbf{q}_{goal}) = 0$
- $\mathbf{U}_{att}(\mathbf{q})$ differentiable for all $\mathbf{q}$

$$\begin{aligned}\vec{F}_{att}(\mathbf{q}) = -\nabla\mathbf{U}_{att}(\mathbf{q}) &= -\nabla\frac{1}{2}\xi\rho^2_{goal}(\mathbf{q}) \\ &= -\frac{1}{2}\xi\nabla\rho^2_{goal}(\mathbf{q}) \\ &= -\frac{1}{2}\xi(2\rho_{goal}(\mathbf{q}))\nabla\rho_{goal}(\mathbf{q})\end{aligned}$$

The Gradient $\nabla\rho_{goal}(\mathbf{q})$

Recall: $\rho_{goal}(\mathbf{q}) = \|\mathbf{q} - \mathbf{q}_{goal}\| = \left(\Sigma_i(x_i - x_{g_i})^2\right)^{1/2}$,
where $\mathbf{q} = (x_1, \ldots, x_n)$ and $\mathbf{q}_{goal} = (x_{g_1}, \ldots, x_{g_n})$

$$\begin{aligned}
\nabla\rho_{goal}(\mathbf{q}) &= \nabla\left(\sum_i (x_i - x_{g_i})^2\right)^{1/2} \\
&= \frac{1}{2}\left(\sum_i (x_i - x_{g_i})^2\right)^{-1/2} \nabla\left(\sum_i (x_i - x_{g_i})^2\right) \\
&= \frac{1}{2}\left(\sum_i (x_i - x_{g_i})^2\right)^{-1/2} (2(x_1 - x_{g_1}), \ldots, 2(x_n - x_{g_n})) \\
&= \frac{(x_1, \ldots, x_n) - (x_{g_1}, \ldots, x_{g_n})}{\left(\Sigma_i(x_i - x_{g_i})^2\right)^{1/2}} \\
&= \frac{\mathbf{q} - \mathbf{q}_{goal}}{\|\mathbf{q} - \mathbf{q}_{goal}\|} \quad = \frac{\mathbf{q} - \mathbf{q}_{goal}}{\rho_{goal}(\mathbf{q})}
\end{aligned}$$

So, $-\nabla\rho_{goal}(\mathbf{q})$ is a unit vector directed toward $\mathbf{q}_{goal}$ from $\mathbf{q}$

Thus, since $-\nabla\mathbf{U}_{att}(\mathbf{q}) = -\frac{1}{2}\xi(2\rho_{goal}(\mathbf{q}))\nabla\rho_{goal}(\mathbf{q})$, we get:

$$\vec{F}_{att}(\mathbf{q}) = -\nabla\mathbf{U}_{att}(\mathbf{q}) = -\xi(\mathbf{q} - \mathbf{q}_{goal})$$

- $\vec{F}_{att}(\mathbf{q})$ is a vector directed toward $\mathbf{q}_{goal}$ with magnitude linearly related to the distance from $\mathbf{q}$ to $\mathbf{q}_{goal}$
- $\vec{F}_{att}(\mathbf{q})$ converges linearly to zero as $\mathbf{q}$ approaches $\mathbf{q}_{goal}$ – good for stability
- $\vec{F}_{att}(\mathbf{q})$ grows without bound as $\mathbf{q}$ moves away from $\mathbf{q}_{goal}$ – not so good

Conic Well Attractive Potential

Idea: Use a 'conic well' to keep $\vec{F}_{att}(\mathbf{q})$ bounded

- $\mathbf{U}_{att}(\mathbf{q}) = \xi \rho_{goal}(\mathbf{q})$
- $\vec{F}_{att}(\mathbf{q}) = -\nabla \mathbf{U}_{att}(\mathbf{q}) = -\xi \frac{(\mathbf{q}-\mathbf{q}_{goal})}{\|\mathbf{q}-\mathbf{q}_{goal}\|}$
- $\vec{F}_{att}(\mathbf{q})$ is a unit vector (constant magnitude) directed towards $\mathbf{q}_{goal}$ everywhere except $\mathbf{q} = \mathbf{q}_{goal}$
- $\mathbf{U}_{att}$ is singular at the goal – not stable (might cause oscillations)

Better (?) Idea: A hybrid method with parabolic and conic wells

$$\mathbf{U}_{att}(\mathbf{q}) = \begin{cases} \frac{1}{2}\xi \rho^2_{goal}(\mathbf{q}) & \text{if } \rho_{goal}(\mathbf{q}) \leq d \\ \\ d\xi \rho_{goal}(\mathbf{q}) & \text{if } \rho_{goal}(\mathbf{q}) > d \end{cases}$$

and

$$\vec{F}_{att}(\mathbf{q}) = \begin{cases} -\xi(\mathbf{q} - \mathbf{q}_{goal}) & \text{if } \|\mathbf{q} - \mathbf{q}_{goal}\| \leq d \\ \\ -d\xi \frac{(\mathbf{q}-\mathbf{q}_{goal})}{\|\mathbf{q}-\mathbf{q}_{goal}\|} & \text{if } \|\mathbf{q} - \mathbf{q}_{goal}\| > d \end{cases}$$

The Repulsive Potential

Basic Idea: $\mathcal{A}$ should be repelled from obstacles

- never want to let $\mathcal{A}$ 'hit' an obstacle
- if $\mathcal{A}$ is far from obstacle, don't want obstacle to affect $\mathcal{A}$'s motion

<u>One Choice for $\mathbf{U}_{rep}$</u>:

$$\mathbf{U}_{rep}(\mathbf{q}) = \begin{cases} \frac{1}{2}\eta\left(\frac{1}{\rho(\mathbf{q})} - \frac{1}{\rho_0}\right) & \text{if } \rho(\mathbf{q}) \leq \rho_0 \\ \\ 0 & \text{if } \rho(\mathbf{q}) > \rho_0 \end{cases}$$

where

- $\rho(\mathbf{q})$ is minimum distance from $\mathcal{CB}$ to $\mathbf{q}$, i.e., $\rho(\mathbf{q}) = \min_{\mathbf{q}' \in \mathcal{CB}} \|\mathbf{q} - \mathbf{q}'\|$
- η is a positive scaling factor
- ρ_0 is a positive constant – *distance of influence*

So, as $\mathbf{q}$ approaches $\mathcal{CB}$, $\mathbf{U}_{rep}(\mathbf{q})$ approaches ∞

The Repulsive Force $\vec{F}_{rep}(\mathbf{q}) = -\nabla \mathbf{U}_{rep}(\mathbf{q})$ for convex $\mathcal{CB}$

(unrealistic) Assumption: $\mathcal{CB}$ is a single convex region

$$\begin{aligned}
\vec{F}_{rep}(\mathbf{q}) &= -\nabla \mathbf{U}_{rep}(\mathbf{q}) \\
&= -\nabla \left(\frac{1}{2}\eta \left(\frac{1}{\rho(\mathbf{q})} - \frac{1}{\rho_0} \right)^2 \right) \\
&= -\frac{1}{2}\eta \nabla \left(\frac{1}{\rho(\mathbf{q})} - \frac{1}{\rho_0} \right)^2 \\
&= -\eta \left(\frac{1}{\rho(\mathbf{q})} - \frac{1}{\rho_0} \right) \nabla \left(\frac{1}{\rho(\mathbf{q})} - \frac{1}{\rho_0} \right) \\
&= -\eta \left(\frac{1}{\rho(\mathbf{q})} - \frac{1}{\rho_0} \right) (-1) \left(\frac{1}{\rho^2(\mathbf{q})} \right) \nabla \rho(\mathbf{q}) \\
&= \eta \left(\frac{1}{\rho(\mathbf{q})} - \frac{1}{\rho_0} \right) \left(\frac{1}{\rho^2(\mathbf{q})} \right) \nabla \rho(\mathbf{q})
\end{aligned}$$

Let $\mathbf{q}_c$ be *unique* configuration in $\mathcal{CB}$ closest to $\mathbf{q}$, i.e., $\rho(\mathbf{q}) = \|\mathbf{q} - \mathbf{q}_c\|$

Then, $\nabla \rho(\mathbf{q})$ is unit vector directed away from $\mathcal{CB}$ along the line passing through $\mathbf{q}_c$ and $\mathbf{q}$

$$\nabla \rho(\mathbf{q}) = \frac{\mathbf{q} - \mathbf{q}_c}{\|\mathbf{q} - \mathbf{q}_c\|}$$

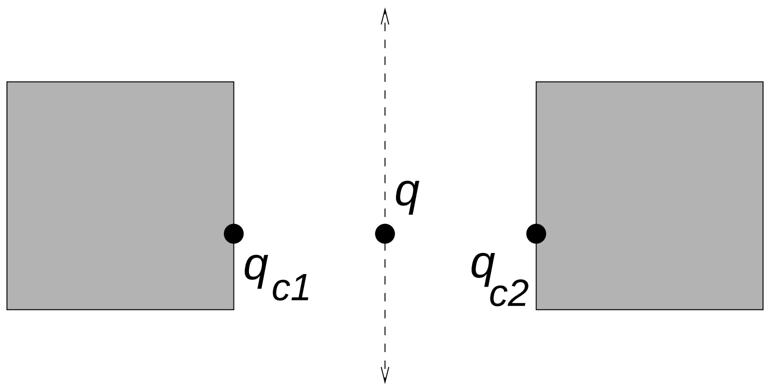
so

$$\vec{F}_{rep}(\mathbf{q}) = \eta \left(\frac{1}{\rho(\mathbf{q})} - \frac{1}{\rho_0} \right) \left(\frac{1}{\rho^2(\mathbf{q})} \right) \frac{\mathbf{q} - \mathbf{q}_c}{\|\mathbf{q} - \mathbf{q}_c\|}$$

The Repulsive Force for non-convex $\mathcal{CB}$

If $\mathcal{CB}$ is not convex, $\rho(\mathbf{q})$ is differentiable everywhere except for at configurations $\mathbf{q}$ which have more than one closest point $\mathbf{q}_c$ in $\mathcal{CB}$

In general, the set of closest points $\mathbf{q}_c$ to $\mathbf{q}$ is $n-1$-dimensional (where n is the dimension of $\mathcal{C}$)



Note: $\vec{F}_{rep}(\mathbf{q})$ exists on both sides of this line, but points in different directions (towards line) and could result in paths that oscillate

Usual Approach: Break $\mathcal{CB}$ (or $\mathcal{B}$) into convex pieces

- associate repulsive field with each convex piece
- final repulsive field is the sum
- potential trouble that several small $\mathcal{CB}_i$ may combine to generate a repulsive force greater than would be produced by a single larger obstacle
  – can weight fields according to size of $\mathcal{CB}_i$

Notes on Repulsive Fields

on designing $\mathbf{U}_{rep}$

- can select different η and ρ_0 for each obstacle region – ρ_0 small for $\mathcal{CB}_i$ close to goal (or else repulsive force may keep us from ever reaching goal)
- if $\mathbf{U}_{rep}(\mathbf{q}_{goal}) \neq 0$, then global minimum of $\mathbf{U}(\mathbf{q})$ is generally not at $\mathbf{q}_{goal}$

on computing $\mathbf{U}_{rep}$

- pretty easy if $\mathcal{CB}$ is polygonal or polyhedral
- really hard for arbitrary shaped $\mathcal{CB}$
- can try to break $\mathcal{CB}$ into convex pieces (not necessary polyhedral) – then can use iterative, numerical methods to find closest boundary points

Gradient Descent Potential Guided Planning

Using a potential field (attractive and repulsive) for path planning...

GRADIENT DESCENT PLANNING

input: $\mathbf{q}_{init}$, $\mathbf{q}_{goal}$, $\mathbf{U}(\mathbf{q}) = \mathbf{U}_{att}(\mathbf{q}) + \mathbf{U}_{rep}(\mathbf{q})$, and $\vec{F}(\mathbf{q}) = -\nabla\mathbf{U}(\mathbf{q})$

output: a path connecting $\mathbf{q}_{init}$ and $\mathbf{q}_{goal}$

1. let $\mathbf{q}_0 = \mathbf{q}_{init}$, $i = 0$
2. if $\mathbf{q}_i \neq \mathbf{q}_{goal}$
   then $\mathbf{q}_{i+1} = \mathbf{q}_i + \delta_i \frac{\vec{F}(\mathbf{q})}{\|\vec{F}(\mathbf{q})\|}$ {`take a step of size` δ_i `in direction` $\vec{F}(\mathbf{q})$}
   else stop
3. set $i = i + 1$ and goto step 2

Notes/Difficulties/Issues:

- originally proposed and well-suited for on-line planning where obstacles are 'sensed' during motion execution [Khatib 86], [Koditschek 89]
- also called 'Steepest Descent' or 'Depth-First' Planning
- local minima are a major problem – recognizing and escaping ...
  – heuristics for escaping [Donald 84, Donald 87]
- step size δ_i
  - δ_i should be small enough so that no collision is possible when moving along straight-line segment $\mathbf{q}_i, \mathbf{q}_{i+1}$ in C-space, e.g., set δ_i smaller than minimum (current) distance to $\mathcal{CB}$
  - δ_i shouldn't let us overshoot goal
- how to evaluate $\rho(\mathbf{q})$ and $\nabla\rho(\mathbf{q})$ which appear in the equations for $\vec{F}(\mathbf{q})$, i.e., in finding the closest point of $\mathcal{CB}$ to current configuration $\mathbf{q}$