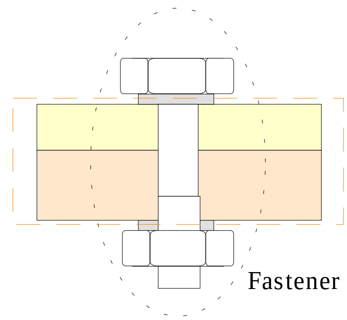
Joint Stiffness

A typical joint is composed of two components, the fastener and the members. Each has a stiffness that contributes to the overall stiffness of the joint, and are identified in the figure.



Fastener Stiffness

The fastener generally consists of two distinct sections, the threaded and the unthreaded. The overall stiffness of the fastener is determined by modeling each segment as a spring and noting that they are springs in series, so the overall stiffness is determined from the relation

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

Definitions of the fastener length (*L*), the grip length (L_G), overall thread length (L_T), threaded length within the grip region (l_t), fastener diameter (*d*), washer thickness (*t* or t_w), etc. is given in Table 8-7 (p 462) of the text for through-the-thickness fasteners (side *a*) and non-through-the thickness fasteners (side *b*). For fasteners with long threaded regions the stiffness of the threaded region is

$$k_T = \frac{A_t E}{l_t}$$

The stiffness in the unthreaded region of a fastener is given by

$$k_d = \frac{A_d E}{l_d}$$

In these two expressions: A_t = tensile-stress area of the fastener (Tables 8-1 and 8.2 on pp 448 and 449)
l_t = length of the threaded portion of the grip
A_d = major-diameter cross-sectional area of the fastener
l_d = length of the unthreaded portion in the grip
E = elastic modulus of the bolt material

Therefore, the estimated effective stiffness of the fastener in the clamped region of the joint (k_b) is given by

$$\frac{1}{k_b} = \frac{1}{k_T} + \frac{1}{k_d} = \frac{k_T + k_d}{k_T k_d} = \frac{\frac{A_t E}{l_t} + \frac{A_d E}{l_d}}{\left(\frac{A_t E}{l_t}\right)\left(\frac{A_d E}{l_d}\right)} = \frac{\frac{A_t l_d E + A_d l_t E}{l_t l_d}}{\frac{A_t A_d E^2}{l_t l_d}} = \frac{A_t l_d E + A_d l_t E}{A_t A_d E^2} = \frac{A_t l_d + A_d l_t}{A_t A_d E}$$

As a result

$$k_b = \frac{A_t A_d E}{A_t l_d + A_d l_t}$$

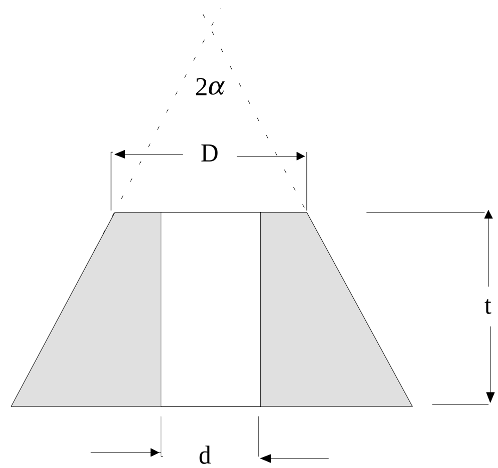
Member Stiffness

The stiffness of the member is determined by considering the effective spring rate of each component of the member. For joints with multiple members, this accomplished by considering a number of springs in series. For a joint consisting of *n* members we have

$$\frac{1}{k_m} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n}$$

The general expression for the stiffness of a frustrum is given by

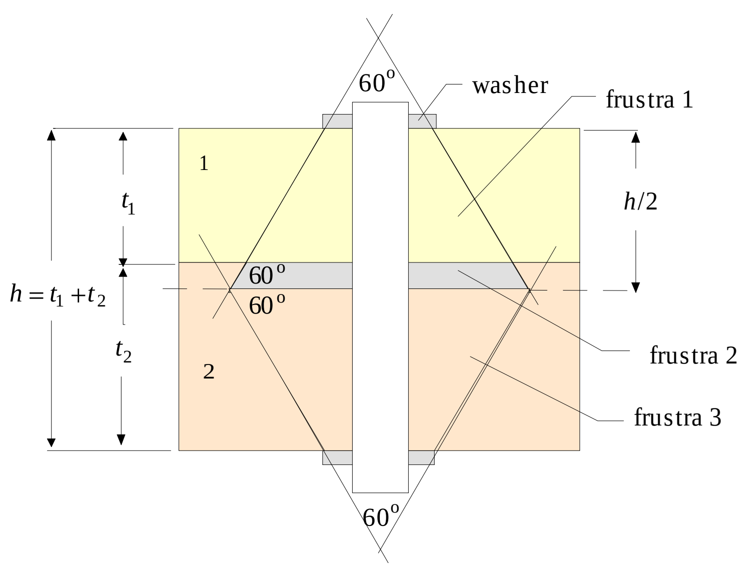
$$k = \frac{\pi E d \tan\alpha}{\ln \dfrac{(2t\tan\alpha + D - d)(D + d)}{(2t\tan\alpha + D + d)(D - d)}}$$



It is generally accepted that $\alpha = 30^{\circ}$ should be used for most cases. As a result

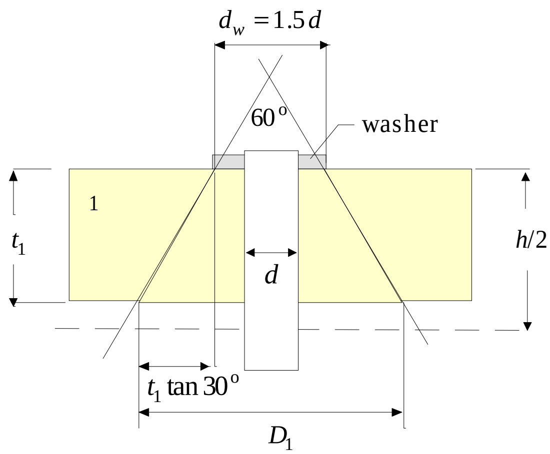
$$k = \frac{0.577\pi E d}{\ln \dfrac{(1.15t + D - d)(D + d)}{(1.15t + D + d)(D - d)}}$$

This equation is solved for each frustrum in a joint and the results are combined in order to define the member stiffness k_m. For example, assume a joint with two members. The total thickness of the joint is $h = t_1 + t_2$. In order to identify the number of frustra in the joint;



1. Locate the mid-plane of the joint at $h/2$.
2. Construct two equilateral triangles, one above the mid-plane and one blow the mid-plane.
3. The washer diameter is generally taken as a reference point of the top and bottom surfaces of the joint from which construction of the triangles is begun. A general rule of thumb is that the washer diameter is 1.5 times the bolt diameter $d_w = 1.5d$.
4. Count the number of frustra that exist. In the case shown to the right, there are three. Frustra 1 is above the mid-plane, frustra 3 is below the mid-plane, and frustra 2 is the region where the two equilateral triangles overlap. The dimensions from each of these regions is used to define the stiffness of the individual frustra, and subsequently the stiffness of the joint.

For the upper frustra (frustra 1) the stiffness is defined from the relation

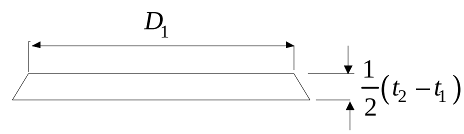


$$k_1 = \frac{0.577\pi E d}{\ln \dfrac{(1.15t + D - d)(D + d)}{(1.15t + D + d)(D - d)}}$$

where D is the diameter at the top of the frustra ($D = d_w = 1.5d$), d is the bolt diameter and t is the thickness of the upper frustra ($t = t_1$). Using these diameters and thickness, and noting that the elastic modulus for frustra 1 is $E = E_1$

$$k_1 = \frac{0.577\pi E_1 d}{\ln \dfrac{(1.15t_1 + d_w - d)(d_w + d)}{(1.15t_1 + d_w + d)(d_w - d)}} = \frac{0.577\pi E_1 d}{\ln \dfrac{(1.15t_1 + 0.5d)(2.5d)}{(1.15t_1 + 2.5d)(0.5d)}}$$

We note that the diameter D_1 shown in the figure is $D_1 = 2t_1 \tan 30^o + d_w = 1.15t_1 + 1.5d$. This is used as D in the computation of the stiffness for the middle frustra (frustra 2). The thickness of the middle frustra is $t = (t_2 - t_1)/2$. For the middle frustra we use $E = E_2$, so the stiffness of the middle frustra is given by

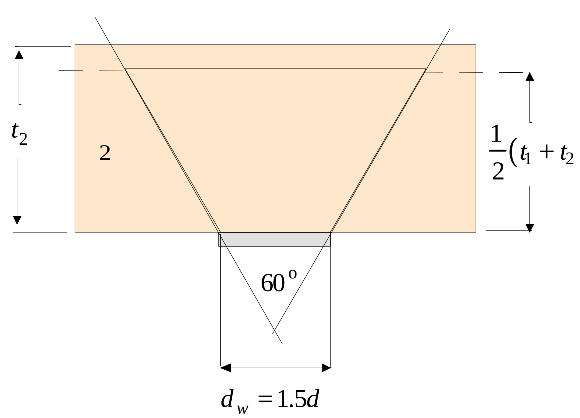


$$k_2 = \frac{0.577\pi E_2 d}{\ln \dfrac{\left(1.15\left[(t_2 - t_1)/2\right] + D_1 - d\right)(D_1 + d)}{\left(1.15\left[(t_2 - t_1)/2\right] + D_1 + d\right)(D_1 - d)}}$$

Using $D_1 = 1.15t_1 + 1.5d$ results in

$$k_2 = \frac{0.577\pi E_2 d}{\ln \dfrac{\left(0.575(t_2 - t_1) + 1.15t_1 + 1.5d - d\right)(1.15t_1 + 1.5d + d)}{\left(0.575(t_2 - t_1) + 1.15t_1 + 1.5d + d\right)(1.15t_1 + 1.5d - d)}} = \frac{0.577\pi E_2 d}{\ln \dfrac{\left(0.575(t_2 + t_1) + 0.5d\right)(1.15t_1 + 2.5d)}{\left(0.575(t_2 + t_1) + 2.5d\right)(1.15t_1 + 0.5d)}}$$

The lower frustra (frustra 3) is handled in a similar manner, so that



$$k_3 = \frac{0.577\pi E d}{\ln \dfrac{(1.15t + D - d)(D + d)}{(1.15t + D + d)(D - d)}}$$

where D is the diameter at the bottom of the frustra ($D = d_w = 1.5d$), d is the bolt diameter and t is the thickness of the bottom frustra ($t = (t_1 + t_2)/2$). Using these diameters and thickness, and noting that the elastic modulus for frustra 1 is $E = E_2$, the stiffness of the bottom frustra is

$$k_3 = \frac{0.577\pi E_2 d}{\ln \frac{\left(1.15\left(t_1+t_2\right)/2+1.5d-d\right)\left(1.5d+d\right)}{\left(1.15\left(t_1+t_2\right)/2+1.5d+d\right)\left(1.5d-d\right)}} = \frac{0.577\pi E_2 d}{\ln \frac{\left(0.575\left(t_1+t_2\right)+0.5d\right)\left(2.5d\right)}{\left(0.575\left(t_1+t_2\right)+2.5d\right)\left(0.5d\right)}}$$

The bolt stiffness must also be considered. Assuming a through bolt, Table 8-7 gives

$$k_b = \frac{A_d A_t E}{A_d l_t + A_t l_d}$$

The stiffness of the member (joint) is therefore

$$\frac{1}{k_m} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_b}$$

Assuming a 12-mm (0.012 m) diameter ISO class 8.8 bolt, with $t_1 = 20\ \text{mm}\ (=0.02\ \text{m})$, $t_2 = 25\ \text{mm}\ (=0.025\ \text{m})$, $E_1 = 207\ \text{GPa}$, $E_2 = 113\ \text{GPa}$, and $E_b = 207\ \text{GPa}$ we get the following.

Frustra 1:

$$k_1 = \frac{0.577\pi E_1 d}{\ln \frac{\left(1.15t_1+0.5d\right)\left(2.5d\right)}{\left(1.15t_1+2.5d\right)\left(0.5d\right)}} = \frac{0.577\pi\left(207\times10^9\right)\left(0.012\right)}{\ln \frac{\left(1.15\left(0.020\right)+0.5\left(0.012\right)\right)\left(2.5\left(0.012\right)\right)}{\left(1.15\left(0.020\right)+2.5\left(0.012\right)\right)\left(0.5\left(0.012\right)\right)}}$$

$$k_1 = \frac{4502.7\times10^6}{\ln \frac{0.000870}{0.000318}} = 4473.9\times10^6\ \text{N/m}$$

Sililarly

$$k_2 = \frac{0.577\pi E_2 d}{\ln \frac{\left(0.575\left(t_2+t_1\right)+0.5d\right)\left(1.15t_1+2.5d\right)}{\left(0.575\left(t_2+t_1\right)+2.5d\right)\left(1.15t_1+0.5d\right)}}$$

$$= \frac{0.577\pi\left(113\times10^9\right)\left(0.012\right)}{\ln \frac{\left(0.575\left(0.025+0.020\right)+0.5\left(0.012\right)\right)\left(1.15\left(0.020\right)+2.5\left(0.012\right)\right)}{\left(0.575\left(0.025+0.020\right)+2.5\left(0.012\right)\right)\left(1.15\left(0.020\right)+0.5\left(0.012\right)\right)}} = \frac{2458\times10^6}{\ln \frac{0.0016894}{0.0016202}} = 58770\times10^6\ \text{N/m}$$

For the third frustra we have

$$k_3 = \frac{0.577\pi E_2 d}{\ln \frac{\left(0.575\left(t_1+t_2\right)+0.5d\right)\left(2.5d\right)}{\left(0.575\left(t_1+t_2\right)+2.5d\right)\left(0.5d\right)}} = \frac{0.577\pi\left(113\times10^9\right)\left(0.012\right)}{\ln \frac{\left(0.575\left(0.025+0.020\right)+0.5\left(0.012\right)\right)\left(2.5\left(0.012\right)\right)}{\left(0.575\left(0.025+0.020\right)+2.5\left(0.012\right)\right)\left(0.5\left(0.012\right)\right)}}$$

$$= \frac{2458\times10^6}{\ln \frac{0.0009563}{0.0003353}} = 2344.8\times10^6\ \text{N/m}$$

If we now consider the bolt, we note that for the specified type of bolt used, Table E-3 gives

$H = 10.8$ mm, $L_G = 45$ mm, and $H + L_G = 55.8$ mm

From Table E-17; $L = 60$ mm, $l_d = 60 - 30 = 30$ mm, $l_t = 45 - 30 = 15$ mm, $A_d = \pi\left(12^2 / 4\right) = 113$ mm^2
From Table 8-1; $A_t = 84.3$ mm^2, and using EQ (8-17) with $A_d = 113 \times 10^{-6}$ m^2 and $A_t = 84.3 \times 10^{-6}$ m^2

$$k_b = \frac{113 \times 10^{-6}(84.3 \times 10^{-6})(207 \times 10^9)}{113 \times 10^{-6}(0.015) + 84.3 \times 10^{-6}(0.030)} = 466.8 \times 10^6 \text{ N/m}$$

For the entire joint we therefore have

$$\frac{1}{k_m} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_b} = \frac{1}{4473.9 \times 10^6} + \frac{1}{58770 \times 10^6} + \frac{1}{2344.8 \times 10^6} + \frac{1}{466.8 \times 10^6}$$

$$k_m = 358 \times 10^6 \text{ N/m}$$