Intersection of a Line and a Sphere (or circle)

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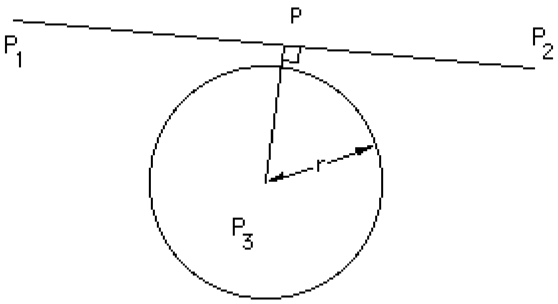
Source code example by: Iebele Abel

http://astronomy.swin.edu.au/~pbourke/geometry/sphereline/



Points **P** (x,y) on a line defined by two points $\mathbf{P}_1$ (x_1,y_1,z_1) and $\mathbf{P}_2$ (x_2,y_2,z_2) is described by

$$\mathbf{P} = \mathbf{P}_1 + u\,(\mathbf{P}_2 - \mathbf{P}_1)$$

or in each coordinate

$$x = x_1 + u\,(x_2 - x_1)$$
$$y = y_1 + u\,(y_2 - y_1)$$
$$z = z_1 + u\,(z_2 - z_1)$$

A sphere centered at $\mathbf{P}_3$ (x_3,y_3,z_3) with radius r is described by

$$(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2 = r^2$$

Substituting the equation of the line into the sphere gives a quadratic equation of the form

$$a\,u^2 + b\,u + c = 0$$

where:

$$a = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$b = 2[\,(x_2 - x_1)(x_1 - x_3) + (y_2 - y_1)(y_1 - y_3) + (z_2 - z_1)(z_1 - z_3)\,]$$

$$c = x_3^2 + y_3^2 + z_3^2 + x_1^2 + y_1^2 + z_1^2 - 2[x_3 x_1 + y_3 y_1 + z_3 z_1] - r^2$$

The solutions to this quadratic are described by

$$\frac{-b \pm \mathrm{sqrt}(b^2 - 4\,a\,c)}{2a}$$

The exact behaviour is determined by the expression within the square root

$$b * b - 4 * a * c$$

- If this is less than 0 then the line does not intersect the sphere.
- If it equals 0 then the line is a tangent to the sphere intersecting it at one point, namely at $u = -b/2a$.
- If it is greater then 0 the line intersects the sphere at two points.

To apply this to two dimensions, that is, the intersection of a line and a circle simply remove the z component from the above mathematics.

Line Segment

When dealing with a line segment it may be more efficient to first determine whether the line actually intersects the sphere or circle. This is achieved by noting that the closest point on the line through $\mathbf{P}_1\mathbf{P}_2$ to the point $\mathbf{P}_3$ is along a perpendicular from $\mathbf{P}_3$ to the line. In other words if $\mathbf{P}$ is the closest point on the line then

$$(\mathbf{P}_3 - \mathbf{P}) \text{ dot } (\mathbf{P}_2 - \mathbf{P}_1) = 0$$

Substituting the equation of the line into this

$$[\mathbf{P}_3 - \mathbf{P}_1 - u(\mathbf{P}_2 - \mathbf{P}_1)] \text{ dot } (\mathbf{P}_2 - \mathbf{P}_1) = 0$$

Solving the above for u =

$$\frac{(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1) + (z_3 - z_1)(z_2 - z_1)}{(x_2 - x_1)(x_2 - x_1) + (y_2 - y_1)(y_2 - y_1) + (z_2 - z_1)(z_2 - z_1)}$$

If u is not between 0 and 1 then the closest point is not between $\mathbf{P}_1$ and $\mathbf{P}_2$

Given u, the intersection point can be found, it must also be less than the radius r. If these two tests succeed then the earlier calculation of the actual intersection point can be applied.