The Chow-Kontsevich dilogarithm

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Abstract. Based on a variant of the Kontsevich $1\frac{1}{2}$-logarithm function, we construct a regulator for a curve over the ring of dual numbers of a field of characteristic p. This also leads to an infinitesimal invariant of certain cycles in characteristic p.

1. Introduction

The Shannon entropy function defined by

$$H(x) := -x\log(x) - (1-x)\log(1-x),$$

for $0 < x < 1$, satisfies the fundamental equation of information theory:

(1.0.1) $$H(x) + (1-x)H(\frac{y}{1-x}) = H(y) + (1-y)H(\frac{x}{1-y}).$$

This same equation reappeared in the work of Cathelineau ([2], [3]) which gave an infinitesimal analog of Hilbert's third problem. If k is a field of characteristic 0, let $\beta_2(k)$ be the vector space over k generated by the symbols $\langle x \rangle$, for $x \in k^\times \setminus \{1\}$ with relations generated by

$$\langle x \rangle - \langle y \rangle + (1-x)\langle \frac{y}{1-x} \rangle - (1-y)\langle \frac{x}{1-y} \rangle$$

when $x + y \neq 1$. The map $D : \beta_2(k) \to k \otimes k^\times$ defined on the generators by

$$D(\langle a \rangle) := a \otimes a + (1-a) \otimes (1-a)$$

is an infinitesimal analog of the Dehn invariant. Cathelineau proves that the cokernel of D is isomorphic to Ω^1_k [2, Théorème 1]. The infinitesimal version of the scissors congruence group can be seen as the limit of the hyperbolic scissors congruence group as it approached to the euclidean one ([5],[6]). The observation that in the limit the hyperbolic volume map should approach the euclidean volume map, led Bloch and Esnault to define an additive dilogarithm based on a K-theoretic complex in [1]. A similar dilogarithm map was expected on an infinitesimal scissors congruence group [6]. Letting $k_m := k[t]/(t^m)$, such a dilogarithm function ℓi_2 (§2.2.1), on the Bloch group $B_2(k_m)$ (§2.1) is defined in [10]. Letting $\delta : B_2(k_m) \to \Lambda^2 k_m^\times$ be the Bloch complex (§2.1), it was shown that the infinitesimal part of $ker(\delta)$ is isomorphic to the indecomposable part of $K_3(k_m)/K_3(k)$, and the infinitesimal part of $coker(\delta)$ is isomorphic to $K_2^M(k_m)/K_2^M(k)$, as expected. Moreover, for $m = 2$, the Bloch complex includes Cathelineau's complex $\beta_2(k) \to k \otimes k^\times$ as a subcomplex.

The indecomposable part of $K_3(k_2)/K_3(k)$ is isomorphic to $k \oplus k$, if k is a field of characteristic $p \geq 5$, whereas it is isomorphic to k, if k is of characteristic 0. This suggests that in characteristic p, there might be two independent dilogarithm functions on $B_2(k_2)$. In [8], Kontsevich defined the function

$$\pounds_1(s) := \sum_{1 \leq i \leq p-1} \frac{s^i}{i},$$

which he called the $1\frac{1}{2}$-logarithm since it satisfies the four-term functional equation (1.0.1). This function was modified in [11], to define a characteristic p dilogarithm map $\ell i_2^{(p)}$ (§3), from $B_2(k_2)$ to k. This dilogarithm function $\ell i_2^{(p)}$ together with ℓi_2 was used to compute cohomology of the Bloch complex in characteristic p in [11]. The aim of the present work is to define an infinitesimal

Chow dilogarithm on a curve, which reduces to $\ell i_2^{(p)}$ for the projective line. Next we describe the main theorem.

Assume that k is a field of characteristic $p \geq 5$ and C is a smooth and proper curve over k_2. For each closed point c of C, we fix, once and for all, a closed subscheme $\mathfrak{c}$ of C such that $\mathfrak{c}$ is smooth over k_2 and has support equal to c. Let $\mathscr{P}$ denote the set of all these subschemes $\mathfrak{c}$, and let η be the generic point of C. For a point x of C let $\mathcal{O}_{C,x}$ denote the local ring of C at x. Let $k(C, \mathscr{P})^\times$ denote the subgroup of $\mathcal{O}_{C,\eta}^\times$, which consist of those f in $\mathcal{O}_{C,\eta}^\times$ such that if c is a closed point of C and $\tilde{s} \in \mathcal{O}_{C,c}$ defines the closed subscheme $\mathfrak{c}$ then $f = u\tilde{s}^n$, for some $n \in \mathbb{Z}$ and $u \in \mathcal{O}_{C,c}^\times$.

In §2.3, we modify the construction in characteristic 0 in [12] to obtain the infinitesimal Chow-dilogarithm

$$\rho : \Lambda^3 k(C, \mathscr{P})^\times \to k.$$

However, the main construction of this paper is the map

$$\rho_K : \Lambda^3 k(C, \mathscr{P})^\times \to k$$

which is based on the Kontsevich logarithm and is a purely characteristic p object. This is stated as the main theorem:

Theorem 1.0.1. *Let k be a field of characteristic $p \geq 5$, and C a smooth and projective curve over k_2 and $\mathscr{P}$ and $k(C, \mathscr{P})^\times$ as above. We have a regulator map:*

$$\rho_K : \Lambda^3 k(C, \mathscr{P})^\times \to k,$$

associated to the Kontsevich $1\frac{1}{2}$-logarithm $\pounds_1$, which canonically associates an element in k to a triple of functions on C. When C is the projective line $\mathbb{P}^1_{k_2}$ with coordinate function z, then

$$\rho_K((z - \alpha) \wedge (z - \beta) \wedge (z - \gamma)) = a^p \cdot \pounds_1(s),$$

where $\frac{\gamma - \beta}{\alpha - \beta} = s + as(1 - s)t$, with $a \in k$, $s \in k^\times \setminus \{1\}$.

We call the function ρ_K, the Chow-Kontsevich dilogarithm. The regulators ρ_K and ρ are linearly independent. If t denotes the variable in k_2, rescaling t to $a \cdot t$, results in multiplying ρ with a^3 whereas ρ_K is multiplied with a^p.

One can interpret the existence of ρ_K as part of a strong reciprocity law on curves. Let X be a smooth and projective curve over a field k. Suslin proved that the sum of norms of the residue maps

$$K_n^M(k(X)) \xrightarrow{\oplus_{x \in |X|} \mathrm{res}_x} \oplus_{x \in |X|} K_{n-1}^M(k(x)) \xrightarrow{\oplus_{x \in |X|} N_{k(x)/k}} K_{n-1}^M(k)$$

from the Milnor K-group of the function field of X to those of the closed points of X, is equal to 0 [9]. When $n = 1$, this states that the degree of the divisor of a rational function is 0. When $n = 2$, this is a restatement of Weil reciprocity. We are interested in the case when $n = 3$. Assuming that k is algebraically closed, Goncharov conjectures in [7] a stronger version of this reciprocity law: the sum of the residue maps from the motivic complex of $k(X)$ to that of k is homotopic to 0. In particular, this implies the existence of a map from $\Lambda^3 k(X)^\times$ to the Bloch group $B_2(k)$ (§2.1). Composing with any map from $B_2(k)$ to a group A gives a map from $\Lambda^3 k(X)^\times$ to A.

The analog of Goncharov's conjecture in our main set-up would give us a map $\Lambda^3 k(C, \mathscr{P})^\times \to B_2(k_2)$ (cf. [12, §3.4]). Composing with the characteristic p dilogarithm $\ell i_2^{(p)} : B_2(k_2) \to k$, would give a map from $\Lambda^3 k(C, \mathscr{P})^\times$ to k. The Chow-Kontsevich dilogarithm ρ_K is expected to be this function.

We will also use ρ_K to define an infinitesimal invariant of cycles. Let $k_\infty := k[[t]]$, and $\square_{k_\infty}^n := k_\infty \times_k (\mathbb{P}^1_k \setminus \{1\})^n$. For $0 \leq q, n$, we define a certain subgroup $\underline{z}_f^q(k_\infty, n)$ of cycles of codimension q in $\square_{k_\infty}^n$ (§7.2.1), [12, §4]. For fixed q, these groups form a complex. Based on the construction of the map in Theorem 1.0.1, we define a regulator map

$$\rho_K : \underline{z}_f^2(k_\infty, 3) \to k,$$

which vanishes on boundaries and has the property that if two cycles are congruent modulo (t^2) then they have the same image under ρ_K (Theorem 7.2.3).

Outline. In §2.1 and §2.2 , we review the definitions of the Bloch group and the Bloch complex, the additive dilogarithm of [10] and the construction of the infinitesimal Chow dilogarithm in the characteristic 0 case [12]. In §2.3, we describe the modifications that are needed in order to carry this construction to characteristic p. In §3, we review the construction of $\ell i_2^{(p)}$, the additive dilogarithm in characteristic p, based on the Kontsevich $1\frac{1}{2}$-logarithm and its relation to K-theory [11]. In §4, we describe the construction of the 1-form $\Omega^{(p)}$ which helps us control different local liftings of a curve. This section handles the split case. In §5, we prove that the residues of the 1-form $\Omega^{(p)}$ are independent of the parametrization: the main result is stated as Proposition 5.4.1, the main step of the proof is Lemma 5.2.1. In §5.5, we use this invariance of residues to define the residue of the 1-form $\Omega^{(p)}$ for a pair of smooth algebras of dimension 1 over k_p, which have isomorphic local rings at their generic points. Lemma 5.5.2 shows that this residue depends only on the reduction of this isomorphism to the reduced closed subscheme. In §6, we relate the additive dilogarithm $\ell i_2^{(p)}$ to the residue of $\Omega^{(p)}$ in Proposition 6.0.5. In §7.1, we prove Theorem 1.0.1, which constructs the Chow-Kontsevich dilogarithm and use this construction in §7.2 to define an infinitesimal invariant of cycles in $\underline{z}_f^2(k_\infty, 3)$.

Notation. Let A be a ring and I be an ideal of A. If $a \in A$, we let $a|_I \in A/I$ denote the reduction of a modulo I. If $p = (a, b, c) \in A^3$ is a triple of elements in A, we write $p|_I$ for $(a|_I, b|_I, c|_I) \in (A/I)^3$. If the ideal is $I = (t^m)$, we also write $a|_{t^m}$ instead of $a|_{(t^m)}$. If $\alpha : X \to Y$ is any function and $p = (a, b, c) \in X \times X \times X$, we abuse the notation and write $\alpha(p)$ for $(\alpha(a), \alpha(b), \alpha(c))$.

For a ring R, we let $R_\infty := R[[t]]$ be the formal power series ring over R and $R_m := R[t]/(t^m)$, the truncated polynomial ring over R of modulus m. If R is a $\mathbb{Q}$-algebra then the exponential map is defined as $e^\alpha := \sum_{0 \le n} \frac{\alpha^n}{n!}$ for $\alpha \in (t) \subseteq R_\infty$. The same formula defines a map for $\alpha \in (t) \subseteq R_m$. On the other hand, if R is a ring of characteristic p, even though there is no such an exponential map on R_∞, we define the modified version $\underline{e}^\alpha := \sum_{0 \le n < p} \frac{\alpha^n}{n!}$, for $\alpha \in (t) \subseteq R_\infty$. This gives a map from tR_∞ to R_∞. We denote the induced map from tR_p to R_p with the same symbol.

2. The infinitesimal Chow dilogarithm

2.1. **Definitions of the complexes.** If R is any ring, we let $R^\flat := \{r \in R | r(1 - r) \in R^\times\}$. The Bloch group $B_2(R)$ is defined to be the quotient of $\mathbb{Z}[R^\flat]$ by the subgroup generated by

$$(2.1.1) \qquad [x] - [y] + [y/x] - [(1 - x^{-1})/(1 - y^{-1})] + [(1 - x)/(1 - y)],$$

for all $x, y \in R^\flat$ with $x - y \in R^\times$. Let $\delta : B_2(R) \to \Lambda^2 R^\times$ denote the map which is defined on the generators by $\delta([x]) := (1 - x) \wedge x$. The complex obtained by putting $B_2(R)$ in degree 1 and $\Lambda^2 R^\times$ in degree 2 is called *the Bloch complex of weight 2*:

$$B_2(R) \xrightarrow{\ \delta\ } \Lambda^2 R^\times.$$

2.2. **The characteristic 0 case.** In this section, we review the theory in characteristic 0. We refer the reader to [10] for the details on the additive dilogarithm and to [12] and [13], for the details on the infinitesimal Chow dilogarithm.

For a ring R, let $R_\infty := R[[t]]$ denote the ring of formal power series over R. If R is a $\mathbb{Q}$-algebra, let $\log : (1 + tR_\infty)^\times \to R_\infty$ denote the logarithm given by $\log(1 + z) := \sum_{1 \le n} (-1)^{n+1} \frac{z^n}{n}$, for $z \in tR_\infty$. Let $\log^\circ : R_\infty^\times \to R_\infty$, be the branch of the logarithm associated to the splitting of $R_\infty \to R$ corresponding to the inclusion $R \hookrightarrow R_\infty$. In other words, $\log^\circ$ is defined as $\log^\circ(\alpha) := \log(\frac{\alpha}{\alpha(0)})$, for $\alpha \in R_\infty^\times$. For $q = \sum_{0 \le i} q_i t^i \in R_\infty$, we define $t_i(q) := q_i$. For $\alpha \in R_\infty^\times$, we let $\ell i(\alpha) := t_i(\log^\circ(\alpha))$.

2.2.1. *Additive dilogarithm.* If R is a $\mathbb{Q}$-algebra, and $m \ge 2$, we defined maps

$$\ell i_{m,r} : B_2(R_m) \to R,$$

for each $m < r < 2m$. In this paper, we will only use this map for $m = 2$ and $r = 3$ and with the notation ℓi_2 instead of $\ell i_{2,3}$. Hence ℓi_2 defines a map

$$\ell i_2 : B_2(R_2) \to R.$$

Explicitly, this map is given by

(2.2.1) $$\ell i_2([s+at]) = -\frac{a^3}{2s^2(1-s)^2}.$$

This can also be described using the map $\delta : B_2(R_\infty) \to \Lambda^2 R_\infty^\times$ as follows. If we let

$$\ell_2 \wedge \ell_1 : \Lambda^2 R_\infty^\times \to R$$

be given by $(\ell_2 \wedge \ell_1)(a \wedge b) := \ell_2(a)\ell_1(b) - \ell_2(b)\ell_1(a)$ then the map $(\ell_2 \wedge \ell_1) \circ \delta$ factors through the canonical surjection $B_2(R_\infty) \to B_2(R_2)$:

$$
\begin{array}{ccc}
B_2(R_\infty) & \xrightarrow{\ \delta\ } & \Lambda^2 R_\infty^\times \\
\downarrow & & \downarrow{\scriptstyle \ell_2 \wedge \ell_1} \\
B_2(R_2) & \xrightarrow{\ \ell i_2\ } & R,
\end{array}
$$

and, abusing the notation, we write

(2.2.2) $$\ell i_2 = (\ell_2 \wedge \ell_1) \circ \delta.$$

In the following, we will also use the notation $\ell := \ell_2 \wedge \ell_1$ and hence write

(2.2.3) $$\ell i_2 = \ell \circ \delta.$$

2.2.2. *Infinitesimal Chow dilogarithm.* In this section, we review the infinitesimal Chow dilogarithm when the base field k is of characteristic 0. We refer the reader to [12] and [13] for the details.

Definition 2.2.1. Let $\mathcal{S}$ be a smooth algebra of relative dimension 1 over k_m, with $m \in \mathbb{N} \cup \{\infty\}$, such that the reduction $\underline{\mathcal{S}}$ of $\mathcal{S}$ modulo (t) is a discrete valuation ring. Let c be the closed point of $\mathrm{Spec}(\mathcal{S})$. We say that a closed subscheme $\mathfrak{c}$ of $\mathrm{Spec}(\mathcal{S})$ is a *smooth lifting* of c, if $\mathfrak{c}$ is smooth over k_m and is supported on c. If $\tilde{s}$ is an element of $\mathcal{S}$ such that its reduction in $\underline{\mathcal{S}}$ is a uniformizer, we also call $\tilde{s}$ a *uniformizer*. If η is the generic point of $\mathrm{Spec}(\mathcal{S})$ and $\mathcal{S}_\eta$ is its local ring at η, then we let

$$(\mathcal{S}, \tilde{s})^\times := \{\alpha \in \mathcal{S}_\eta^\times \,|\, \alpha = u\tilde{s}^n, \text{ for some } u \in \mathcal{S}^\times \text{ and } n \in \mathbb{Z}\}.$$

If the closed subscheme on $\mathcal{S}$ defined by $\tilde{s}$ is $\mathfrak{c}$, then we say that $\tilde{s}$ is a *uniformizer for* $\mathfrak{c}$. In this case, we also write $(\mathcal{S}, \mathfrak{c})^\times := (\mathcal{S}, \tilde{s})^\times$.

Note that since any two uniformizers for $\mathfrak{c}$ differ by multiplication by an element in $\mathcal{S}^\times$, the definition of $(\mathcal{S}, \mathfrak{c})^\times$ is independent of the choice of the uniformizer. We say that an element $\alpha \in \mathcal{S}_\eta$ is *good with respect to* $\mathfrak{c}$ or equivalently is $\mathfrak{c}$-*good* if $\alpha \in (\mathcal{S}, \mathfrak{c})^\times$. If C is a smooth curve over k_m and c is a closed point of C, the analogous notions are defined by taking $\mathcal{S}$ as the local ring of C at c.

Let $\mathcal{R}$ be any smooth algebra of relative dimension 1 over k_m, whose reduction $\underline{\mathcal{R}}$ is not necessarily a discrete valuation ring, and c be a closed point of $\mathrm{Spec}(\mathcal{R})$. We define a smooth lifting of c to $\mathrm{Spec}(\mathcal{R})$ as a smooth lifting of c to $\mathrm{Spec}(\mathcal{R}_c)$.

Definition 2.2.2. Suppose that $\mathcal{R}$ is a smooth algebra of relative dimension 1 over k_m, with $m \in \mathbb{N} \cup \{\infty\}$. Fix a smooth lifting $\mathfrak{c}$ of every closed point c of $\mathrm{Spec}(\mathcal{R})$ and denote the set of these liftings by $\mathscr{P}$. We let

$$(\mathcal{R}, \mathscr{P})^\times := \bigcap_{c \in |\mathrm{Spec}(\mathcal{R})|} (\mathcal{R}_c, \mathfrak{c})^\times$$

and $(\mathcal{R}, \mathscr{P})^\flat := \{f \in (\mathcal{R}, \mathscr{P})^\times \,|\, 1 - f \in (\mathcal{R}, \mathscr{P})^\times\}$. We define $B_2(\mathcal{R}, \mathscr{P})$ to be the quotient of $\mathbb{Z}[(\mathcal{R}, \mathscr{P})^\flat]$ by the group generated by the five term relations (2.1.1) associated to pairs x and y in $(\mathcal{R}, \mathscr{P})^\flat$ with the property that $x - y \in (\mathcal{R}, \mathscr{P})^\times$.

Let $k(c)$ denote the residue field of c and $k(\mathfrak{c})$ denote the ring of regular functions on the affine scheme $\mathfrak{c}$. We have a map

$$\delta : B_2(\mathcal{R}, \mathscr{P}) \to \Lambda^2(\mathcal{R}, \mathscr{P})^\times,$$

as above, whose value on the generator $[x]$ is given by $\delta([x]) = (1-x) \wedge x$. This induces a map

$$\delta : B_2(\mathcal{R}, \mathscr{P}) \otimes (\mathcal{R}, \mathscr{P})^\times \to \Lambda^3(\mathcal{R}, \mathscr{P})^\times,$$

which sends $[x] \otimes y$ to $\delta([x]) \wedge y$. Abusing the notation, we denote this map also with the symbol δ. For each $\mathfrak{c}$, we have residue maps

$$res_{\mathfrak{c}} : B_2(\mathcal{R}, \mathscr{P}) \otimes (\mathcal{R}, \mathscr{P})^\times \to B_2(k(\mathfrak{c})),$$

and

$$res_{\mathfrak{c}} : \Lambda^3(\mathcal{R}, \mathscr{P})^\times \to \Lambda^2 k(\mathfrak{c})^\times,$$

which give a commutative diagram:

$$
\begin{array}{ccc}
B_2(\mathcal{R}, \mathscr{P}) \otimes (\mathcal{R}, \mathscr{P})^\times & \xrightarrow{\delta} & \Lambda^3(\mathcal{R}, \mathscr{P})^\times \\
\downarrow{res_{\mathfrak{c}}} & & \downarrow{res_{\mathfrak{c}}} \\
B_2(k(\mathfrak{c})) & \xrightarrow{\delta} & \Lambda^2 k(\mathfrak{c})^\times.
\end{array}
$$

Suppose that C is a smooth and projective curve over k_m and $|C|$, the set of its closed points and let η be its generic point. For each $c \in |C|$, fix a smooth lifting $\mathfrak{c}$ of c to C and let $\mathscr{P}$ denote the set of all of these liftings. For an open subset U of C, let $\mathscr{P}|_U$ denote the set of those $\mathfrak{c} \in \mathscr{P}$ such that $\mathfrak{c} \subseteq U$.

Let $(\mathcal{O}_C, \underline{\mathscr{P}})^\times$ (resp. $(\mathcal{O}_C, \underline{\mathscr{P}})^\flat$) be the sheaf on C such that

$$(\mathcal{O}_C, \underline{\mathscr{P}})^\times(U) = (\mathcal{O}_C(U), \mathscr{P}|_U)^\times \quad (\text{resp. } (\mathcal{O}_C, \underline{\mathscr{P}})^\flat(U) = (\mathcal{O}_C(U), \mathscr{P}|_U)^\flat)$$

for each affine open subset U of C. Similarly, let $B_2(\mathcal{O}_C, \underline{\mathscr{P}})$ be the sheaf associated to the presheaf whose sections on such U are $B_2(\mathcal{O}_C(U), \mathscr{P}|_U)$.

This gives us a complex $\mathscr{C}(C, \mathscr{P})$ of sheaves on C which are concentrated in degrees 2 and 3:

(2.2.4) $$B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^\times \to \oplus_{c \in |C|} i_{c*}(B_2(k(\mathfrak{c}))) \oplus \Lambda^3(\mathcal{O}_C, \underline{\mathscr{P}})^\times,$$

where $i_c : c \to C$ is the inclusion map. Let $k(C)^\times := \mathcal{O}_{C,\eta}^\times$ denote group of units of the local ring of C at its generic point η and $k(C, \mathscr{P})^\times := \Gamma(C, (\mathcal{O}_C, \underline{\mathscr{P}})^\times) \subseteq k(C)^\times$ denote those which are also $\mathscr{P}$-good.

When $m = 2$, in other words when C/k_2 is a smooth and projective curve, the infinitesimal Chow dilogarithm ρ is a map $\rho : \Lambda^3 k(C, \mathscr{P})^\times \to k$.

Suppose that $\mathcal{A}/k_\infty$ is a smooth algebra over k_∞ of relative dimension 1, and $\mathscr{P}$ is a set of smooth liftings of closed points of $\mathrm{Spec}(\mathcal{A})$ as above. For $\mathfrak{c} \in \mathscr{P}$, recall that $k(\mathfrak{c})$ denotes the ring of functions on the affine scheme $\mathfrak{c}$. If k' denotes the residue field of $\mathfrak{c}$, there is a unique isomorphism of k_∞-algebras between $k(\mathfrak{c})$ and k'_∞ which is the identity map modulo (t). Therefore we can identify $k(\mathfrak{c})$ and k'_∞. In particular, this gives us a canonical map $\ell : \Lambda^2 k(\mathfrak{c})^\times \to k'$, which corresponds to $\ell = \ell_2 \wedge \ell_1 : \Lambda^2(k'_\infty)^\times \to k'$ via the above identification $k(\mathfrak{c}) = k'_\infty$.

If $\tilde{f}, \tilde{g}, \tilde{h} \in (\mathcal{A}, \mathscr{P})^\times$ and $\mathfrak{c} \in \mathscr{P}$ then $res_{\mathfrak{c}}(\tilde{f} \wedge \tilde{g} \wedge \tilde{h}) \in \Lambda^2 k(\mathfrak{c})^\times$. Applying ℓ to this object we obtain an element in k' and taking its trace to k, we obtain

$$\mathrm{Tr}_k(\ell(res_{\mathfrak{c}}(\tilde{f} \wedge \tilde{g} \wedge \tilde{h}))) \in k.$$

This term will be essential in defining the local contribution to the Chow dilogarithm.

Suppose that C/k_2 has a global lifting $\tilde{C}/k_\infty$. Namely, $\tilde{C}$ is a smooth and projective curve over k_∞ together with a fixed isomorphism between $\tilde{C} \times_{k_\infty} k_2$ and C. Let $\tilde{\mathscr{P}}$ be a set of smooth liftings for each point of $c \in |C|$ to $\tilde{C}$ which reduce to the liftings in $\mathscr{P}$ modulo (t^2). Suppose further that $\tilde{f}, \tilde{g}, \tilde{h} \in k(\tilde{C}, \tilde{\mathscr{P}})^\times$ reduce to f, g and h modulo (t^2). Then

(2.2.5) $$\rho(f \wedge g \wedge h) = \sum_{c \in |C|} \mathrm{Tr}_k(\ell(res_{\tilde{c}}(\tilde{f} \wedge \tilde{g} \wedge \tilde{h}))),$$

where $\tilde{c} \in \tilde{\mathscr{P}}$ denotes the lifting of c.

In general, such global liftings do not exist. Even when they exist, we do not know, a priori, that (2.2.5) is independent of the choice of the liftings. Our method to define ρ in general, is to choose local liftings of the curve and the functions and also to choose a generic lifting of the curve and the functions and then to measure the defects between the local liftings and the generic lifting. This defect between the different liftings will be measured by the residue of a 1-form. Next we describe this in detail.

We start with the following data. Let $\tilde{\mathcal{R}}$ and $\hat{\mathcal{R}}$ are smooth k_∞-algebras of relative dimension one, and χ an isomorphism $\chi : \tilde{\mathcal{R}}/(t^2) \xrightarrow{\sim} \hat{\mathcal{R}}/(t^2)$ of k_2-algebras. Suppose further that we a have a triple $\tilde{p} := (\tilde{f}, \tilde{g}, \tilde{h})$ of functions in $\tilde{\mathcal{R}}^\times$ and a triple $\hat{p} := (\hat{f}, \hat{g}, \hat{h})$ of functions in $\hat{\mathcal{R}}^\times$, such that $\chi(\tilde{p}|_{t^2}) = \hat{p}|_{t^2}$. To this data, we will attach an element $\omega(\tilde{p}, \hat{p}, \chi) \in \Omega^1_{\underline{\hat{\mathcal{R}}}/k}$, where $\underline{\hat{\mathcal{R}}}$ is the reduction modulo (t) of $\hat{\mathcal{R}}$.

We will proceed as follows. Let $\overline{\chi} : \tilde{\mathcal{R}} \xrightarrow{\sim} \hat{\mathcal{R}}$ be any k_∞-algebra isomorphism which reduces to χ modulo (t^2) and $\varphi : \underline{\hat{\mathcal{R}}} \to \hat{\mathcal{R}}$ be any splitting of the canonical projection. Denote by $\overline{\varphi}$ the corresponding isomorphism $\underline{\hat{\mathcal{R}}}[[t]] \xrightarrow{\sim} \hat{\mathcal{R}}$ of k_∞-algebras, which extend φ. Then we define:

$$\omega(\tilde{p}, \hat{p}, \chi) := \Omega(\overline{\varphi}^{-1}(\overline{\chi}(\tilde{p})), \overline{\varphi}^{-1}(\hat{p})),$$

with Ω as below.

Let $\tilde{q} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ and $\hat{q} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$, with $\tilde{y}_i, \hat{y}_i \in \underline{\hat{\mathcal{R}}}[[t]]^\times$, and $\hat{y}_i - \tilde{y}_i \in (t^2)$, for all $1 \le i \le 3$. There are unique $\alpha_{0i} \in \underline{\hat{\mathcal{R}}}^\times$ and $\alpha_{1i}, \hat{\alpha}_{ji}, \tilde{\alpha}_{ji} \in \underline{\hat{\mathcal{R}}}$, for $1 \le i \le 3$ and $2 \le j$ such that

$$\hat{y}_i = \alpha_{0i} e^{t\alpha_{1i} + t^2 \hat{\alpha}_{2i} + \cdots}$$

and

$$\tilde{y}_i = \alpha_{0i} e^{t\alpha_{1i} + t^2 \tilde{\alpha}_{2i} + \cdots}.$$

We then define

$$\Omega(\tilde{q}, \hat{q}) := \sum_{\sigma \in S_3} (-1)^\sigma \alpha_{1\sigma(1)} (\tilde{\alpha}_{2\sigma(3)} - \hat{\alpha}_{2\sigma(3)}) \cdot d\log(\alpha_{0\sigma(2)}) \in \Omega^1_{\underline{\hat{\mathcal{R}}}/k}.$$

It turns out that the definition of $\omega(\tilde{p}, \hat{p}, \chi)$ is independent of the choices of the lifting $\overline{\chi}$ and the splitting φ. It does depend on the liftings of the triples of functions and on χ, as reflected in the notation.

Suppose that C/k_2 and $\mathscr{P}$ are as above and f, g, h are in $k(C, \mathscr{P})^\times$. We will describe the definition of $\rho(f \wedge g \wedge h) \in k$ below.

First let $p := (f, g, h)$ and let η be the generic point on C. Let $\tilde{\mathcal{A}}$ be a smooth k_∞-algebra together with an isomorphism $\alpha : \tilde{\mathcal{A}}/(t^2) \xrightarrow{\sim} \mathcal{O}_{C,\eta}$. Let $\tilde{p}_\eta$ be a triple of functions in $\tilde{\mathcal{A}}$, whose reductions modulo (t^2) map to the germs p_η of the functions p at η. For each $c \in |C|$, let $\widehat{\mathcal{B}}^\circ_c$ be a smooth k_∞-algebra together with an isomorphism $\tilde{\gamma}_c : \widehat{\mathcal{B}}^\circ_c/(t^2) \xrightarrow{\sim} \hat{\mathcal{O}}_{C,c}$, from the reduction of $\widehat{\mathcal{B}}^\circ_c$ modulo (t^2) to the completion of the local ring of C at c. Let $\mathscr{P}_c = \{\mathfrak{c}\}$ denote the smooth lifting of the point c to C. Let $\mathcal{Q}_c := \tilde{\gamma}_c^*(\mathscr{P}_c)$ denote the corresponding smooth lifting on $\mathrm{Spec}(\widehat{\mathcal{B}}^\circ_c)$ obtained by transport of structure via the isomorphism $\tilde{\gamma}_c$. If p_c denotes the triple of functions obtained by taking the germs of the functions in p at the point c then $p_c \in ((\hat{\mathcal{O}}_{C,c}, \mathscr{P}_c)^\times)^3$. With the notation above, we have $q_c := \tilde{\gamma}_c^{-1}(p_c) \in ((\widehat{\mathcal{B}}^\circ_c/(t^2), \mathcal{Q}_c)^\times)^3$. Let $\tilde{\mathcal{Q}}_c$ be a smooth lifting to $\mathrm{Spec}(\widehat{\mathcal{B}}^\circ_c)$ which reduces to $\mathcal{Q}_c$ modulo (t^2). Let $\tilde{q}_c \in ((\widehat{\mathcal{B}}^\circ_c, \tilde{\mathcal{Q}}_c)^\times)^3$ which reduces to q_c modulo (t^2). If $\tilde{q}_c = (\alpha, \beta, \gamma)$, and $\tilde{\mathcal{Q}}_c = \{\tilde{\mathfrak{c}}\}$ then $res_{\tilde{\mathfrak{c}}}(\alpha \wedge \beta \wedge \gamma) \in \Lambda^2 k(\tilde{\mathfrak{c}})^\times$. We denote the last expression $res_{\tilde{\mathfrak{c}}}(\alpha \wedge \beta \wedge \gamma)$ by $res_c(\tilde{q}_c)$ in order to simplify the notation.

If we let $\tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_c$ denote the isomorphism obtained by composing the completion of α at c and the localization of the inverse of $\tilde{\gamma}_c$ at the generic point then the value of ρ on p is given by:

$$(2.2.6) \qquad \rho(p) := \sum_{c \in |C|} \mathrm{Tr}_k(\ell(res_c(\tilde{q}_c)) + res_c\omega(\tilde{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_c)).$$

Here the term $\omega(\tilde{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_c)$ should be interpreted in the following sense. The map $\tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_c$ is an isomorphism form the completion $\hat{\tilde{\mathcal{A}}}_c/(t^2)$ of $\tilde{\mathcal{A}}/(t^2)$ at c to the localization $\widehat{\mathcal{B}}^\circ_{c,\eta}/(t^2)$ of

$\widetilde{\mathcal{B}}_c^\circ/(t^2)$ at the generic point η. The images of the triples of functions $\tilde{p}_\eta$ and $\tilde{q}_c$ in the completions and localisations give liftings to $\hat{\tilde{A}}_c$ and to $\widetilde{\mathcal{B}}_{c,\eta}^\circ$. This is precisely the set-up in which we can use ω to obtain a 1-form $\omega(\tilde{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_c)$ on $\widetilde{\mathcal{B}}_{c,\eta}^\circ/(t)$. The residue of this 1-form at the closed point of $\widetilde{\mathcal{B}}_c^\circ/(t)$ is an element in the residue field of c.

It is the main theorem of [12] that the above sum is finite and is independent of all the choices involved. Moreover, ρ induces a map from $\Lambda^3 k(C, \mathscr{P})^\times$ to k.

2.3. **Modifications in the characteristic p case.** Suppose that R is a ring of characteristic p. We cannot define a logarithm map from $(1 + rR_\infty)$ to R_∞ since the power series expression of the logarithm map has denominators. On the other hand, there is a well-defined logarithm homomorphism

$$\log : (1 + tR_m)^\times \to R_m$$

given by $\log(1 + z) = \sum_{1 \le n < p} (-1)^{n+1} \frac{z^n}{n}$, for $z \in tR_m$, when $m \le p$. Similarly, the map $\log^\circ : R_m^\times \to R_m$ is defined by $\log^\circ(\alpha) = \log(\frac{\alpha}{\alpha(0)})$ and is a homomorphism. As above, the maps $\ell_i : R_m^\times \to R$ are defined by the formula $\ell_i(\alpha) = t_i(\log^\circ(\alpha))$, for $1 \le i < m$.

We can define a map $\ell i_2 : B_2(R_2) \to R$ using the formula (2.2.1), when $p > 2$. More explicitly, the map $\ell := \ell_2 \wedge \ell_1 : \Lambda^2 R_3^\times \to R$ is used to define $\ell i_2 : B_2(R_2) \to R$ by the following commutative diagram

$$
\begin{array}{ccc}
B_2(R_3) & \xrightarrow{\delta} & \Lambda^2 R_3^\times \\
\downarrow & & \downarrow{\ell_2 \wedge \ell_1} \\
B_2(R_2) & \xrightarrow{\ell i_2} & R,
\end{array}
$$

exactly analogous to (2.2.2).

If we assume now that C/k_2 is a smooth and proper curve, where k is a field of characteristic $p > 2$ then the construction in the previous section carries over in this case to give a map $\rho : \Lambda^3 k(C, \mathscr{P})^\times \to k$. In the construction, we need to make the following modifications. In the characteristic 0 case we choose liftings $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}_c^\circ$ which are smooth over k_∞. In the characteristic p case, we will choose these liftings to be smooth k_3-algebras. Also in the definition of ω, we will start with smooth k_3-algebras $\hat{\mathcal{R}}$ and $\tilde{\mathcal{R}}$. Then $\overline{\chi}$ will be a morphism of k_3-algebras and $\overline{\varphi}$ will be an isomorphism from $\hat{\mathcal{R}}_3$ to $\hat{\mathcal{R}}$, where $\hat{\underline{\mathcal{R}}} := \hat{\mathcal{R}}/(t)$. Finally, we note that the definition of Ω works as in characteristic 0. In the definition of Ω, in order to obtain the coefficients α_{1i}, $\hat{\alpha}_{2i}$ and $\tilde{\alpha}_{2i}$, we are essentially using ℓ_1 and ℓ_2, which also makes sense on $\hat{\underline{\mathcal{R}}}_3$.

3. THE ADDITIVE DILOGARITHM IN CHARACTERISTIC p

From now, we assume that $p \ge 5$ throughout the paper. When R is a ring of characteristic p, the additive dilogarithm $\ell i_2 : B_2(R) \to R$ does not tell the whole story. There is another function, which is of characteristic p in nature that completes the picture. Such a function was constructed in [11]. We describe this function in this section.

The construction is based on the $1\frac{1}{2}$-logarithm defined by Kontsevich in [8]. Let us first recall this function which is defined as:

$$\pounds_1(s) = \sum_{1 \le i \le p-1} \frac{s^i}{i},$$

for $s \in R$. If we define

$$\ell i_2^{(p)}([s + \alpha t]) := \frac{\alpha^p}{s^p(1-s)^p} \pounds_1(s) = \frac{\alpha^p}{s^p(1-s)^p} \sum_{1 \le i \le p-1} \frac{s}{i},$$

for $s + \alpha t \in R_2^\flat$, we deduce that $\ell i_2^{(p)}$ induces a map:

$$\ell i_2^{(p)} : B_2(R_2) \to R$$

using the functional equations for $\mathcal{L}_1$ [11]. We note that with the notation of [11], we have $\ell i_2^{(p)} = \mathfrak{L}\mathfrak{i}_2^p$. Because of the context, it is not possible to confuse the notation $\ell i_2^{(p)}$ for the one for the p-adic dilogarithm.

Similar to ℓi_2 the map $\ell i_2^{(p)}$ can be expressed in terms of the differential in the Bloch complex. Namely, the diagram:

$$
\begin{array}{ccc}
B_2(R_p) & \xrightarrow{\ \delta\ } & \Lambda^2 R_p^\times \\
\downarrow & & \Big\downarrow {\scriptstyle \frac{1}{2}\sum_{1\le i<p} i\cdot \ell_{p-i}\wedge \ell_i} \\
B_2(R_2) & \xrightarrow{\ \ell i_2^{(p)}\ } & R,
\end{array}
$$

commutes, allowing us to write

(3.0.1) $$\ell i_2^{(p)} = \Big(\frac{1}{2}\sum_{1\le i<p} i\cdot \ell_{p-i}\wedge \ell_i\Big)\circ \delta.$$

We put $\ell^{(p)} := \frac{1}{2}\sum_{1\le i<p} i\cdot \ell_{p-i}\wedge \ell_i$, and rewrite the above expression as

(3.0.2) $$\ell i_2^{(p)} = \ell^{(p)}\circ \delta.$$

The above expression should be thought of as analogous to the expressions (2.2.2) and (2.2.3) which express ℓi_2 in terms of δ and ℓ and used to construct the infinitesimal Chow dilogarithm above. Similarly, the expression (3.0.2) will be used to define a Chow-Kontsevich dilogarithm.

We mentioned in the beginning of this section that, in characteristic p, the additive dilogarithm does not tell the whole story. More precisely, that one does not get an injective regulator map if one restricts ℓi_2 to the appropriate part of the K-group. Below is a justification that together with $\ell i_2^{(p)}$, they suffice [11]:

Theorem 3.0.1. *Let k be the algebraic closure of $\mathbb{F}_p$. The direct sum $\ell i_2 \oplus \ell i_2^{(p)}$ induces an isomorphism*
$$K_3^\circ(k_2) \to B_2^\circ(k_2) \to k \oplus k$$
when restricted to the infinitesimal part of $K_3(k_2)$.

We expect the analog of the above theorem to hold for any field k of characteristic $p \ge 5$. In the general case, however, one would get an isomorphism with indecomposable part of $K_3(k_2)^\circ$.

4. The comparison 1-form for the Chow-Kontsevich dilogarithm

One of the most essential steps in defining the infinitesimal Chow dilogarithm was the construction of a 1-form that compares the different choices of liftings for the parameters. More precisely, this 1-form Ω has the property that for two different choices of liftings, the difference of the ℓ values of their residues can be expressed in terms of the residue of Ω [12, Proposition 2.4.4]. In characteristic p, we will construct a similar 1-form, which will denote with $\Omega^{(p)}$, that will have this property with $\ell^{(p)}$ replaced with ℓ in the last sentence.

4.1. **The definition of $\Omega^{(p)}$ on $\Lambda^3 R_\infty^\times$.** Suppose that R is a ring of characteristic p. We define $\Omega^{(p)}$ on $\Lambda^3 R_\infty^\times$ as the unique map
$$\Omega^{(p)} : \Lambda^3 R_\infty^\times \to \Omega_R^1$$
which satisfies the following properties (i)-(vi).

 (i) $\Omega^{(p)}(u \wedge v \wedge w) = 0$, if $u \in 1 + (t^p)R_\infty$.
This last expression implies that the map $\Omega^{(p)}$ descends to give a map from $\Lambda^3 R_p^\times$. Every element of $R_p^\times$ is represented by products of terms of the form f with $f \in R^\times \subseteq R_p^\times$ and terms of the form $\underline{e}^{\alpha t^a} \in R_p^\times$, with $\alpha \in R$ and $0 < a < p$. It is sufficient to determine $\Omega^{(p)}$ on the elements of the form f and $\underline{e}^{\alpha t^a}$. We proceed to do this.

 (ii) $\Omega^{(p)}(f \wedge g \wedge h) = \Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge g \wedge h) = 0$, for $f, g, h \in R^\times$ and $\alpha \in R$, and $0 < a < p$.

 (iii) $\Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge h) = 0$, if $a + b \ne p$; and $\Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge h) = \alpha b\beta \frac{dh}{h}$ for $h \in R^\times$ and $\alpha, \beta \in R$, and $p > a > b > 0$, with $a + b = p$.

(iv) $\Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge \underline{e}^{\gamma t^c}) = 0$, if $a + b + c \neq p$.

(v) $\Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge \underline{e}^{\gamma t^c}) = \alpha(b\beta \cdot d\gamma - c\gamma \cdot d\beta) \in \Omega^1_R$, if $a + b + c = p$ and $a > b \geq c > 0$. Here, $d : R \to \Omega^1_R$ denotes the canonical derivation.

(vi) $\Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge \underline{e}^{\gamma t^c}) = \gamma(a\alpha \cdot d\beta - b\beta \cdot d\alpha) \in \Omega^1_R$, if $a + b + c = p$ and $a = b > c > 0$.

We can rephrase the above definition much more concisely by slightly abusing the notation. First note that in the above, the contribution coming from the term $h \in R^\times$ is $\frac{dh}{h}$. If we abuse the notation and write $h = \underline{e}^\gamma$, then $\frac{dh}{h} = d\gamma$ and this expression resembles the contributions coming from other terms of the form $\underline{e}^{\alpha t^a}$ with $0 < a$.

We can then replace the conditions (ii)-(vi) above with the following equivalent formulation. For $p > a \geq b \geq c \geq 0$, we define $\Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge \underline{e}^{\gamma t^c})$ as:

$(ii)'$ 0, if $a + b + c \neq p$

$(iii)'$ $\alpha(b\beta \cdot d\gamma - c\gamma \cdot d\beta)$, if $a + b + c = p$, and $a > b \geq c \geq 0$

$(iv)'$ $\gamma(a\alpha \cdot d\beta - b\beta \cdot d\alpha)$, if $a + b + c = p$, and $a = b > c \geq 0$.

Note that in the above expression, if $c = 0$ even though the expression γ does not make sense we set $c\gamma := 0$. We emphasize that the above definition covers all the possibilities since $p \geq 5$.

4.2. **The definition of $\Omega^{(p)}$ for a split algebra.** If A is a ring and I is an ideal of A, we let $(A, I)^\times := \{(a, b) | a, b \in A^\times, \ a - b \in I\}$, and let $\pi_i : (A, I)^\times \to A^\times$, for $i = 1, 2$ denote the two projections. We define a map

$$\tilde{\Omega}^{(p)} : \Lambda^3(R_p, (t))^\times \to \Omega^1_R.$$

Let $s : \Lambda^3(R_p, (t))^\times \to \Lambda^3 R_p^\times$, denote the map given by the difference $s := \Lambda^3 \pi_1 - \Lambda^3 \pi_2$. Then we define $\tilde{\Omega}^{(p)} := \Omega^{(p)} \circ s$ as the composition of s and $\Omega^{(p)}$.

5. The invariance of the residue of $\Omega^{(p)}$ with respect to reparametrization

In order to generalize the definition of $\Omega^{(p)}$ to certain rings without a prescribed set of coordinate functions, we need a certain invariance property for $\Omega^{(p)}$. The aim of this section is to describe and prove this invariance property.

5.1. **An elementary formula for an infinitesimal automorphism of $k((s))_\infty$.** Suppose that σ is the k_∞-automorphism of $k((s))_\infty$ such that $\sigma(s) = s + xt^w$ with $w \geq 1$ and $x \in k((s))$. We first do an elementary computation to express the images of $\sigma(f)$ and $\sigma(e^{\alpha t^a})$ in $k((s))_p$, where $f \in k((s))^\times$ and $\alpha \in k((s))$ and $a \geq 1$, in terms of the logarithmic derivatives of f and the derivatives of α.

The formulas in question do not have p-torsion when they are reduced modulo (t^p) as above. In order to prove the formulas, it is easier to prove them first when k is replaced with a ring R which does not have $\mathbb{Z}$-torsion and then deduce the result for k by using an appropriate map $R \to k$. The advantage of using R is that here one can use the Taylor expansion formula, which has denominators, after passing to $R_\mathbb{Q}$. We start with a lemma which is a restatement of the Taylor expansion formula.

Lemma 5.1.1. *Suppose that R is a ring without $\mathbb{Z}$-torsion and σ is the unique continuous R_∞-automorphism of $R((s))_\infty$ such that $\sigma(s) = s + xt^w$, with $x \in R((s))$ and $w \geq 1$. Then for $f \in R((s))$ we have*

$$\sigma(f) = \sum_{0 \leq i} \frac{(xt^w)^i f^{(i)}}{i!} \in R((s))[[t]] \subseteq R_\mathbb{Q}((s))[[t]].$$

Here, f' is defined by $df = f'ds$ and $f^{(i)} := (f^{(i-1)})'$ with $f^{(0)} := f$, for $i \geq 1$.

Proof. By using R-linearity and continuity, we reduce to the case when $f(s) = s^n$ for some $n \in \mathbb{Z}$. By using the map that sends t to xt^w, we reduce to the case where $\sigma(s) = s + t$. By the naturality of the formula, it suffices to check it for $R = \mathbb{R}$. Let $f(s) = s^n$ for some $n \in \mathbb{Z}$. If we first fix s to be a value near 1 and vary t near 0 and expand $f(s + t)$ near s, the Taylor expansion formula gives $f(s + t) = \sum_{0 \leq i} \frac{f^{(i)}(s)t^i}{i!}$. Now letting s vary, we realize that both sides are analytic functions of s and t near 1 and 0 and we have the formula we are looking for between

two analytic functions. The result then follows by identifying the analytic functions with their power series expansions. $\qquad\square$

Lemma 5.1.2. *Suppose that R and σ are as above. Then, for $1 \leq a$ and $\alpha \in R((s))$,*

$$\sigma(e^{\alpha t^a}) = e^{\sum_{0 \leq i} \frac{x^i \alpha^{(i)}}{i!} t^{a+iw}} \in R_{\mathbb{Q}}((s))_\infty;$$

and, for $f \in R((s)))^\times$,

$$\sigma(f) = f e^{\sum_{1 \leq i} \frac{x^i (f'/f)^{(i-1)}}{i!} t^{iw}} \in R_{\mathbb{Q}}((s))_\infty.$$

Proof. Note that for the proof, we can replace R with $R_{\mathbb{Q}}$ and assume that R is a $\mathbb{Q}$-algebra. If $1 \leq a$ and $\alpha \in R((s))$ then by the previous lemma we have $\sigma(\alpha) = \sum_{0 \leq i} \frac{(xt^w)^i \alpha^{(i)}}{i!}$, which implies that

$$\sigma(e^{\alpha t^a}) = e^{\sigma(\alpha) t^a} = e^{\sum_{0 \leq i} \frac{x^i \alpha^{(i)}}{i!} t^{a+iw}}.$$

On the other hand, for $f \in R((s))^\times$, both sides of the formula for $\sigma(f)$ above are compatible with multiplication. Therefore, it is enough to prove the formula for $f = s$, for $f \in R^\times$ or for $f \in 1 + sR[[s]]$. The formula clearly holds for $f \in R^\times$ since $\sigma(f) = f$ and $f' = 0$. In case $f \in 1 + sR[[s]]$, then letting $\alpha := \log(f) \in R[[s]]$, we have $f = e^\alpha$. Using the previous lemma we obtain

$$\sigma(f) = e^{\sigma(\alpha)} = e^{\sum_{0 \leq i} \frac{x^i \alpha^{(i)}}{i!} t^{iw}} = f e^{\sum_{1 \leq i} \frac{x^i (f'/f)^{(i-1)}}{i!} t^{iw}}.$$

For $f = s$, we have

$$\sigma(s) = s + xt^w = s(1 + \frac{xt^w}{s}) = se^{\log(1 + \frac{xt^w}{s})} = se^{\sum_{1 \leq i} \frac{(-1)^{i-1}}{i}(\frac{xt^w}{s})^i} = se^{\sum_{1 \leq i} \frac{x^i(1/s)^{(i-1)}}{i!} t^{iw}}.$$

$\qquad\square$

For a ring R on which $(p-1)!$ is invertible and for $\alpha \in tR_\infty$, recall that we put $\underline{e}^\alpha := \sum_{0 \leq i < p} \frac{\alpha^i}{i!}$. Next we consider the case when k is a field of characteristic p.

Lemma 5.1.3. *Let σ be the automorphism of the k_∞ algebra $k((s))_\infty$, which is the identity map modulo (t) and has the property that $\sigma(s) = s + xt^w$, with $w \geq 1$ and $x \in k((s))$, then*

$$\sigma(\underline{e}^{\alpha t^a}) = \prod_{0 \leq i < p} \underline{e}^{\frac{x^i \alpha^{(i)}}{i!} t^{a+iw}}$$

in $k((s))_p$, for $1 \leq a$ and $\alpha \in k((s))$; and

$$\sigma(f) = f \prod_{1 \leq i < p} \underline{e}^{\frac{x^i (f'/f)^{(i-1)}}{i!} t^{iw}}$$

in $k((s))_p$, for $f \in k((s))^\times$.

Proof. Suppose that we are given x, f and α as in the statement of the lemma. We can choose a ring R without $\mathbb{Z}$-torsion and on which $(p-1)!$ is invertible such that there is a map $R \to k$, and there is $\tilde{x}, \tilde{f} \in R((s))^\times$ and $\tilde{\alpha} \in R((s))$, which map to x, f and α. We first apply the previous lemma to $\tilde{f}$ and $\underline{e}^{\tilde{\alpha} t^a}$, and the automorphism given by $\tilde{\sigma}(s) = s + \tilde{x}t^w$. If we then reduce the expression modulo (t^p), notice that the expression is in $R((s))_p$, and then take the image under the map from R to k, the lemma follows. $\qquad\square$

Remark. As in §4.1 above, if we slightly abuse the notation, the images under σ of $f \in R((s))^\times$ and of $e^{\alpha t^a}$ with $\alpha \in R((s)))$ and $1 \leq a$ can be expressed using a single formula. Namely, suppose that we allow the notation $f = e^{\alpha t^a}$, with $a = 0$. Then we can express both of the formulas in Lemma 5.1.2 as

$$\sigma(e^{\alpha t^a}) = e^{\sum_{0 \leq i} \frac{x^i \alpha^{(i)}}{i!} t^{a+iw}} \in R_{\mathbb{Q}}((s))_\infty,$$

for $a \geq 0$. Note that if $a = 0$, this formula reads:

$$\sigma(f) = \sigma(e^\alpha) = e^{\sum_{0 \leq i} \frac{x^i \alpha^{(i)}}{i!} t^{iw}} = e^\alpha e^{\sum_{1 \leq i} \frac{x^i \alpha^{(i)}}{i!} t^{iw}} = f e^{\sum_{1 \leq i} \frac{x^i (f'/f)^{(i-1)}}{i!} t^{iw}},$$

if we think of α as $\log(f)$ and hence use the convention that $\alpha^{(i)} := (f'/f)^{(i-1)}$ for $i \geq 1$.

In exactly the same manner, both of the formulas in Lemma 5.1.3 can be expressed as

$$\sigma(\underline{e}^{\alpha t^a}) = \prod_{0 \le i < p} \underline{e}^{\frac{x^i \alpha^{(i)}}{i!} t^{a+iw}} \in k((s))_p,$$

for $0 \le a$.

5.2. The effect of reparametrization on $\Omega^{(p)}$.

In order to prove the invariance with respect to reparametrization, we will start with the lemma below which deals with the most basic infinitesimal automorphism. In fact, these automorphisms will generate all the automorphisms that we are interested in.

Lemma 5.2.1. *Suppose that k is a field of characteristic p, and σ is the k_∞-automorphism of $k((s))_\infty$ given by $\sigma(s) = s + xt^w$, with $1 \le w$ and $x \in k((s))$. Then*

$$(5.2.1) \qquad \Omega^{(p)}\big(\sigma(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge \underline{e}^{\gamma t^c})\big) - \Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge \underline{e}^{\gamma t^c}) \in \Omega^1_{k((s))/k}$$

is an exact form for $a, b, c \ge 0$ and $\alpha, \beta, \gamma \in k((s))$.

Note that in this case of characteristic p dilogarithm, we do not need to assume that $w \ge 2$ as we had to in characteristic 0 [13].

The expression (5.2.1) is 0, if $w \nmid p - (a+b+c)$ or if $a+b+c = p$. So let us assume that $w | p - (a+b+c)$ and let $q = \frac{p-(a+b+c)}{w} > 0$. We will express the above expression as an exact form using the following identity:

$$(5.2.2)$$

$$\Omega^{(p)}\big(\sigma(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge \underline{e}^{\gamma t^c})\big) - \Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge \underline{e}^{\gamma t^c}) = d\Big(\sum_{\substack{i+j+k=q \\ 0 \le i,j,k}} \frac{x^q}{q} S(a,b,c;i,j,k) \alpha^{(i)} \beta^{(j)} \gamma^{(k)} \Big),$$

where $S(a,b,c;i,j,k)$ is defined as follows. We define

$$S(a,b,c;i,j,k) := \frac{bk - cj}{i!j!k!},$$

if $a + iw > \max\{b + jw, c + kw\}$ or $b + jw = c + kw > a + iw$.

In case, $b + jw > \max\{a + iw, c + kw\}$ or $a + iw = c + kw > b + jw$ then we let

$$S(a,b,c;i,j,k) := -S(b,a,c;j,i,k).$$

In case, $c + kw > \max\{a + iw, b + jw\}$ or $a + iw = b + jw > c + kw$ then we let

$$S(a,b,c;i,j,k) := S(c,a,b;k,i,j).$$

Let us explain how the expression on the right hand side of (5.2.2) in fact makes sense. Note that the expression involves terms of the form $\alpha^{(i)} \beta^{(j)} \gamma^{(k)}$. On the other hand for $k = 0$, we defined $\gamma^{(k)}$ only for $c > 0$. Therefore, we have to make sure that the coefficient $S(a,b,c;i,j,k)$ is 0, if $k = c = 0$. The same statement would be true for $j = b = 0$ and $i = a = 0$, by anti-symmetry. If $k = c = 0$, then $(a + iw) + (b + jw) = p$. Since $p > 2$, we then have $a + iw \ne b + jw$. In case, $a + iw > b + jw$ then $S(a,b,c;i,j,k) = \frac{bk - cj}{i!j!k!} = 0$. In case, $b + jw > a + iw$, the same follows by anti-symmetry.

5.3. The proof of Lemma 5.2.1.

We will instead prove the identity (5.2.2), which gives a more precise statement than the lemma.

5.3.1. *The coefficients of $x^{q-1} x' \alpha^{(i)} \beta^{(j)} \gamma^{(k)}$.* Suppose that $i + j + k = q$. The coefficient of $x^{q-1} x' \alpha^{(i)} \beta^{(j)} \gamma^{(k)}$ in the right side of (5.2.2) is $S(a,b,c;i,j,k)$. We need to check that this is the same as its coefficient in the left side of (5.2.2).

Suppose first that $a + iw > \max\{b + jw, c + kw\}$ or $b + jw = c + kw > a + iw$. The term in (5.2.1) that contributes to this coefficient is

$$\frac{x^i}{i!} \alpha^{(i)} \big((b+jw) \frac{x^j}{j!} \beta^{(j)} \frac{kx^{k-1}x'}{k!} \gamma^{(k)} - (c+kw) \frac{x^k}{k!} \gamma^{(k)} \frac{jx^{j-1}x'}{j!} \beta^{(j)}\big) = x^{q-1} x' \alpha^{(i)} \beta^{(j)} \gamma^{(k)} \frac{bk - cj}{i!j!k!},$$

whose coefficient is precisely $S(a,b,c;i,j,k)$. The other cases follow from this one by the anti-symmetry of $\Omega^{(p)}$ and S.

5.3.2. *The coefficients of $\frac{x^q}{i!j!k!}\alpha^{(i)}\beta^{(j)}\gamma^{(k)}$.* We need to check that on both sides of (5.2.2) the coefficients of $\frac{x^q}{i!j!k!}\alpha^{(i)}\beta^{(j)}\gamma^{(k)}$ are the same. Let us temporarily write $\tilde{a} := a+iw$, $\tilde{b} := b+jw$, and $\tilde{c} := c+kw$ with $i+j+k = q+1$ and let $\tilde{a}' := \tilde{a}-w$, $\tilde{b}' := \tilde{b}-w$ and $\tilde{c}' := \tilde{c}-w$. There are many cases to consider and we will only write down the answer in each case. The computations are routine but tedious and are omitted. Without loss of generality, we will assume that $\tilde{a} \geq \tilde{b} \geq \tilde{c}$.

Case (i). If one of the following cases hold:

$$\tilde{a} > \tilde{a}' > \tilde{b} > \tilde{b}' > \tilde{c} > \tilde{c}', \quad \tilde{a} > \tilde{a}' > \tilde{b} > \tilde{c} > \tilde{b}' > \tilde{c}', \quad \tilde{a} > \tilde{a}' > \tilde{b} > \tilde{b}' = \tilde{c} > \tilde{c}',$$

$$\tilde{a} > \tilde{a}' > \tilde{b} = \tilde{c} > \tilde{b}' = \tilde{c}', \quad \tilde{a} > \tilde{b} = \tilde{c} > \tilde{a}' > \tilde{b}' = \tilde{c}'$$

then the coefficient on the left side of (5.2.2) is

$$i \cdot 0 + (-1)j \cdot (c+kw) + k \cdot (b+jw) = kb - jc$$

and the coefficient on the right side of (5.2.2) is

$$\frac{1}{q}(i(kb-jc) + j(kb-(j-1)c) + k((k-1)b) - jc) = \frac{i+j+k-1}{q}(kb-jc) = kb - jc.$$

Therefore the coefficients on both sides of the identity match. Note that the assumption that the characteristic is p is not needed in this case.

Case (ii). If one of the following cases hold:

$$\tilde{a} > \tilde{b} > \tilde{c} > \tilde{a}' > \tilde{b}' > \tilde{c}', \quad \tilde{a} > \tilde{b} > \tilde{a}' > \tilde{c} > \tilde{b}' > \tilde{c}', \quad \tilde{a} > \tilde{b} > \tilde{a}' > \tilde{b}' > \tilde{c} > \tilde{c}',$$

$$\tilde{a} > \tilde{b} > \tilde{a}' > \tilde{b}' = \tilde{c} > \tilde{c}', \quad \tilde{a} > \tilde{b} > \tilde{c} = \tilde{a}' > \tilde{b}' > \tilde{c}'$$

then the coefficient on the left side of (5.2.2) is

$$i(c+kw) + (-1)j \cdot (c+kw) + k \cdot (b+jw) = ic + ikw + bk - jc.$$

Putting first $i = q+1-j-k$ and then noting that $qw = -a-b-c$ in characteristic p this can be rewritten as

$$qc + c - 2jc - 2kc - ka + kw - jkw - k^2w.$$

The coefficient on the right side of (5.2.2) is

$$\frac{1}{q}(i((i-1)c - ka) - j((j-1)c - kb) + k((k-1)b - jc)).$$

Again first putting $q+1-j-k$ instead of i and then using $-w = \frac{a+b+c}{q}$, we see that this expression matches the one above.

Case (iii). If one of the following cases hold:

$$\tilde{a} = \tilde{b} > \tilde{a}' = \tilde{b}' > \tilde{c} > \tilde{c}', \quad \tilde{a} = \tilde{b} > \tilde{c} > \tilde{a}' = \tilde{b}' > \tilde{c}', \quad \tilde{a} = \tilde{b} > \tilde{a}' = \tilde{b}' = \tilde{c} > \tilde{c}',$$

then the coefficient on the left side of (5.2.2) is

$$-i(-(c+kw)) + j(-(c+kw)) + k0 = ic - jc + ikw - jkw.$$

Proceeding exactly as in the previous case, this can be rewritten as

$$qc + c - 2jc - 2kc - ka - kb + kw - 2jkw - k^2w.$$

The coefficient on the right side of (5.2.2) is

$$\frac{1}{q}(-i(ka - (i-1)c) + j(kb - (j-1)c) + k(ja - ib)).$$

This matches the above expression after first we put $i = q+1-j-k$, then replace $a+b+c$ with $-qw$.

Case (iv). If one of the following cases hold:

$$\tilde{a} > \tilde{a}' = \tilde{b} > \tilde{c} > \tilde{b}' > \tilde{c}', \quad \tilde{a} > \tilde{a}' = \tilde{b} > \tilde{b}' > \tilde{c} > \tilde{c}', \quad \tilde{a} > \tilde{a}' = \tilde{b} > \tilde{b}' = \tilde{c} > \tilde{c}',$$

then the coefficient on the left side of (5.2.2) is

$$-i(b+jw) - j(c+kw) + k(b+jw) = -2ib - jc + qb + b - jb - ijw.$$

The coefficient on the right side of (5.2.2) is

$$\frac{1}{q}(i(ja-(i-1)b)-j((j-1)c-kb)+k((k-1)b-jc)).$$

Replacing k with $q+1-i-j$ and proceeding as above we see that this matches the above expression.

Case (v). If $\tilde{a}=\tilde{b}=\tilde{c}>\tilde{a}'=\tilde{b}'=\tilde{c}'$ then the coefficient on the left side of (5.2.2) is 0. In this case, the coefficient on the right side of (5.2.2) is

$$\frac{1}{q}(i(kb-jc)-j(ka-ic)+k(ja-ib))=0.$$

Finally, it can be checked without difficulty that the cases above comprise all the possibilities. Let us at least note that we cannot have the case when $\tilde{a}'=\tilde{b}=\tilde{c}$ since $\tilde{a}'+\tilde{b}+\tilde{c}=p>3$. $\square$

5.4. The invariance of the residues of $\Omega^{(p)}$.

We will next prove the invariance with respect to reparametrization.

Proposition 5.4.1. *Suppose that k is a field of characteristic p, and σ is a continuous k_∞-automorphism of $k((s))_\infty$ which is identity modulo the ideal (t). Then we have the equality of the residues:*

$$res_{s=0}\Omega^{(p)}(\sigma(\underline{e}^{\alpha t^a}\wedge\underline{e}^{\beta t^b}\wedge\underline{e}^{\gamma t^c}))=res_{s=0}\Omega^{(p)}(\underline{e}^{\alpha t^a}\wedge\underline{e}^{\beta t^b}\wedge\underline{e}^{\gamma t^c})\in k,$$

for $a,b,c\geq 0$ and $\alpha,\beta,\gamma\in k((s))$.

Proof. Let us write $\sigma(s)=s+\sum_{1\leq w}x_w t^w$, with $x_w\in k((s)))$. If we let τ be the automorphism of $k((s))_\infty$ given by $\tau(s)=s+\sum_{1\leq w<p}x_w t^w$, since σ and τ are congruent modulo (t^p),

$$\Omega^{(p)}(\sigma(\underline{e}^{\alpha t^a}\wedge\underline{e}^{\beta t^b}\wedge\underline{e}^{\gamma t^c}))=\Omega^{(p)}(\tau(\underline{e}^{\alpha t^a}\wedge\underline{e}^{\beta t^b}\wedge\underline{e}^{\gamma t^c})).$$

Letting similarly τ_w defined by $\tau_w(s)=s+y_w t^w$ for $1\leq w<p$, we notice that τ_w are of the form as in the statement of Lemma 5.2.1. This lemma then implies that for any $q\in\Lambda^3 k((s))_\infty^\times$,

$$res_{s=0}(\Omega^{(p)}(\tau_w(q)))=res_{s=0}(\Omega^{(p)}(q)).$$

Choosing $y_w\in k((s))$ appropriately for $1\leq w<p$, we can write $\tau=\tau_{p-1}\circ\cdots\tau_2\circ\tau_1$ and applying the last equality $(p-1)$-times gives us the equality:

$$res_{s=0}\Omega^{(p)}(\tau(\underline{e}^{\alpha t^a}\wedge\underline{e}^{\beta t^b}\wedge\underline{e}^{\gamma t^c}))=res_{s=0}\Omega^{(p)}(\underline{e}^{\alpha t^a}\wedge\underline{e}^{\beta t^b}\wedge\underline{e}^{\gamma t^c}),$$

from which the proposition follows. $\square$

5.5. The definition of the residue of $\omega^{(p)}$ for a pair of smooth algebras of dimension 1 over k_p.

Using the previous section, we generalize the definition of $\omega^{(p)}$ to pairs of elements in smooth algebras of dimension 1 over k_p with the same reduction. As opposed to the split algebra case, which was considered in §4.2, the general case will essentially depend on Proposition 5.4.1 above.

Let $\mathcal{R}$ be a smooth k_p-algebra of relative dimension 1 as above. Let η be the generic point and x be a closed point of the spectrum of $\mathcal{R}$. Then we define a map

$$res_x\omega^{(p)}:\Lambda^3(\mathcal{R}_\eta,(t))^\times\to k',$$

where k' is the residue field of x, as follows. To ease the notation let us write $\mathcal{S}=\mathcal{R}_\eta$. Since $\mathcal{R}/k_p$ is smooth, there is an isomorphism $\varphi:\mathcal{S}\to\underline{\mathcal{S}}_p$ of k_p-algebras which is the identity map modulo (t). Here $\underline{\mathcal{S}}$ denotes the reduction of $\mathcal{S}$ modulo (t) and $\underline{\mathcal{S}}_p:=\underline{\mathcal{S}}[t]/(t^p)$.

The map φ then induces an isomorphism

$$\Lambda^3\varphi:\Lambda^3(\mathcal{S},(t))^\times\to\Lambda^3(\underline{\mathcal{S}}_p,(t))^\times,$$

by transport of structure, whose composition with

$$\tilde{\Omega}^{(p)}:\Lambda^3(\underline{\mathcal{S}}_p,(t))^\times\to\Omega^1_{\underline{\mathcal{S}}},$$

will be denoted by $\omega_\varphi^{(p)} : \Lambda^3(\mathcal{S}, (t))^\times \to \Omega_{\underline{\mathcal{S}}}^1$. Taking the projection to $\Omega_{\underline{\mathcal{S}}/k}^1$ and taking residue at the closed point x gives us the map $res_x\omega_\varphi^{(p)}$. The next lemma, which shows that the residue is independent of the choice of φ, is essential.

Lemma 5.5.1. *If φ and ψ are two isomorphisms between the k_p-algebras $\mathcal{S}$ and $\underline{\mathcal{S}}_p$ then*

$$res_x\omega_\varphi^{(p)} = res_x\omega_\psi^{(p)}.$$

Proof. By the Cohen structure theorem, if $\hat{\underline{\mathcal{S}}}$ denotes the completion of $\underline{\mathcal{S}}$ at the valuation determined by x, then there is a k-isomorphism $\hat{\underline{\mathcal{S}}} \to k'((s))$. By the smoothness of $\mathcal{S}$ over k_p, we also know that there is a k_p-isomorphism $\mathcal{S} \simeq \underline{\mathcal{S}}_p$ which is identity modulo (t). Therefore, replacing $\underline{\mathcal{S}}$ by $\underline{\mathcal{S}}_p$ and then completing at x, we are reduced to proving the lemma for $\mathcal{S} = k'((s))_p$ and φ and ψ two k_p-automorphisms of $k'((s))_p$ which reduce to the identity automorphism modulo (t). The smoothness assumption implies that k'/k is separable. Since φ and ψ are k-algebra morphisms which reduce to identity modulo (t), the separability of k'/k implies that they are also k'-algebra morphisms.

Suppose that $(\tilde{f}, \hat{f}), (\tilde{g}, \hat{g}), (\tilde{h}, \hat{h}) \in (k'((s))_p, (t))^\times$. Then $res_x\omega_\varphi^{(p)}((\tilde{f}, \hat{f}) \wedge (\tilde{g}, \hat{g}) \wedge (\tilde{h}, \hat{h}))$ is equal to the residue of

(5.5.1) $$\Omega^{(p)}(\varphi(\tilde{f}) \wedge \varphi(\tilde{g}) \wedge \varphi(\tilde{h})) - \Omega^{(p)}(\varphi(\hat{f}) \wedge \varphi(\hat{g}) \wedge \varphi(\hat{h}))$$

at $s = 0$. By Proposition 5.4.1, the residue of

(5.5.2) $$\Omega^{(p)}(\tilde{f} \wedge \tilde{g} \wedge \tilde{h}) - \Omega^{(p)}(\varphi(\tilde{f}) \wedge \varphi(\tilde{g}) \wedge \varphi(\tilde{h}))$$

at $s = 0$, is equal to 0. Since a similar formula is true for the second term in (5.5.1), we can rewrite the residue of this expression as the residue of

$$\Omega^{(p)}(\tilde{f} \wedge \tilde{g} \wedge \tilde{h}) - \Omega^{(p)}(\hat{f} \wedge \hat{g} \wedge \hat{h}),$$

which of course does not depend on φ. $\qquad\square$

Since, as we have seen, $res_x\omega_\varphi^{(p)}$ does not depend on φ, we will denote it by $res_x\omega^{(p)}$. Below, we will need a variant of this construction for pairs of algebras. Next we describe this.

Suppose that A is a ring with an ideal I and B and B' are two A-algebras together with an isomorphism $\chi : B/IB \simeq B'/IB'$ of A-algebras. We let

$$(B, B', \chi)^\times := \{(p, p')|p \in B^\times \text{ and } p' \in B'^\times \text{ s.t. } \chi(p|_I) = p'|_I\},$$

where $p|_I$ denotes the image of p in $(B/IB)^\times$.

Suppose that $\mathcal{R}$ and $\mathcal{R}'$ are smooth k_p-algebras of relative dimension 1, together with a k-isomorphism $\chi : \underline{\mathcal{R}} \to \underline{\mathcal{R}}'$. We identify the spectra of $\mathcal{R}$ and $\mathcal{R}'$ via this isomorphism. Suppose that x is a closed point and η is the generic point of this spectrum. We would like to construct a map

$$res_x\omega_{\chi_\eta}^{(p)} : \Lambda^3(\mathcal{R}_\eta, \mathcal{R}'_\eta, \chi_\eta)^\times \to k',$$

where k' is the residue field of x. Let us put, as above, $\mathcal{S} := \mathcal{R}_\eta$ and $\mathcal{S}' := \mathcal{R}'_\eta$. Since, as above, $\mathcal{S} \simeq \underline{\mathcal{S}}_p$ and $\mathcal{S}' \simeq \underline{\mathcal{S}}'_p$ we can choose an isomorphism $\tilde{\chi}_\eta : \mathcal{S} \to \mathcal{S}'$ which lifts χ_η. The map $\tilde{\chi}_\eta$, induces an isomorphism

$$(\mathcal{S}, \mathcal{S}', \chi_\eta)^\times \xrightarrow{\tilde{\chi}_\eta^*} (\mathcal{S}, (t))^\times .$$

Composing these, we obtain

$$res_x\omega_{\tilde{\chi}_\eta}^{(p)} : \Lambda^3(\mathcal{R}_\eta, \mathcal{R}'_\eta, \chi_\eta)^\times \xrightarrow{\Lambda^3\tilde{\chi}_\eta^*} \Lambda^3(\mathcal{R}_\eta, (t))^\times \xrightarrow{res_x\tilde{\Omega}^{(p)}} k'.$$

Lemma 5.5.2. *The map $res_x\omega_{\tilde{\chi}_\eta}^{(p)}$ does not depend on the choice of the lifting $\tilde{\chi}_\eta$ of χ_η.*

Proof. The proof is exactly the same as that of Lemma 5.5.1. We reduce to the case where $\mathcal{R}_\eta$ and $\mathcal{R}'_\eta$ are both $k'((s))_p$ and $\tilde{\chi}_\eta$ is replaced with φ in the proof of this lemma. Then the result follows from the fact that the residue of (5.5.2) at $s = 0$ is 0. $\qquad\square$

Since this map is independent of the choice of the lifting $\tilde{\chi}_\eta$, we will denote it by $res_x\omega^{(p)}_{\chi_\eta}$. If the map χ_η is clear from the context, we will simply denote it by $res_x\omega^{(p)}$.

The following lemma is an immediate consequence of the definition of $res_x\omega^{(p)}$.

Lemma 5.5.3. *Suppose that $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ are smooth k_p-algebras of relative dimension 1 and $\phi : \underline{\mathcal{R}} \to \underline{\mathcal{R}'}$ and $\psi : \underline{\mathcal{R}'} \to \underline{\mathcal{R}''}$ are k-algebra isomorphisms of their reductions modulo (t). Following the notation above, if $(\mathfrak{q},\mathfrak{q}') \in \Lambda^3(\mathcal{R}_\eta, \mathcal{R}'_\eta, \phi_\eta)^\times$ and $(\mathfrak{q}',\mathfrak{q}'') \in \Lambda^3(\mathcal{R}'_\eta, \mathcal{R}''_\eta, \psi_\eta)^\times$ then*

$$(\mathfrak{q},\mathfrak{q}'') \in \Lambda^3(\mathcal{R}_\eta, \mathcal{R}''_\eta, \psi_\eta \circ \phi_\eta)^\times$$

and

$$res_x\omega^{(p)}(\mathfrak{q},\mathfrak{q}'') = res_x\omega^{(p)}(\mathfrak{q},\mathfrak{q}') + res_x\omega^{(p)}(\mathfrak{q}',\mathfrak{q}'').$$

6. The residue of $\Omega^{(p)}$ in the case of good liftings

In this section, we relate the residues of $\Omega^{(p)}$ to $\ell i_2^{(p)}$ in the case of good liftings, continuing with our assumption that $p \geq 5$. We start with comparing the difference between the values under $\ell^{(p)}$ of the residues of

$$\mathfrak{q}' = (s - xt^w) \wedge \underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b}$$

and

$$\mathfrak{q} = s \wedge \underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b},$$

where $w \geq 2$, $a, b \geq 0$, $x \in k$ and $\alpha, \beta \in k[[s]]$. If $a = 0$ or $b = 0$, we follow the conventions above.

Lemma 6.0.1. *With notation as above, we have*

(6.0.1) $$\ell^{(p)}(res_{s-xt^w}(\mathfrak{q}')) - \ell^{(p)}(res_s(\mathfrak{q})) = res_{s=0}\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}).$$

Proof. We start with noting that

$$res_{s-xt^w}(\mathfrak{q}') = \underline{e}^{\alpha(xt^w)t^a} \wedge \underline{e}^{\beta(xt^w)t^b}$$

and $res_s(\mathfrak{q}) = \underline{e}^{\alpha(0)t^a} \wedge \underline{e}^{\beta(0)t^b}$.

First, suppose that $w \nmid p - (a + b)$ or $p < a + b$. In this case,

$$\ell^{(p)}(res_{s-xt^w}(\mathfrak{q}')) = \ell^{(p)}(res_s(\mathfrak{q})) = 0.$$

On the other hand,

$$\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}) = \Omega^{(p)}(\underline{e}^{-\sum_{1\leq n<p}\frac{1}{n}(\frac{xt^w}{s})^n} \wedge \underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b}) = 0.$$

Similarly, in case $a + b = p$, we have

$$\ell^{(p)}(res_{s-xt^w}(\mathfrak{q}')) = \ell^{(p)}(res_s(\mathfrak{q})) = b\alpha(0)\beta(0)$$

and $\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}) = 0$.

In case, $a = b = 0$ and $w = p$, we similarly have $\ell^{(p)}(res_{s-xt^w}(\mathfrak{q}')) = \ell^{(p)}(res_s(\mathfrak{q})) = 0$ and $\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}) = 0$.

Therefore, from now on, we assume that $w|(p - (a + b))$ and $0 < a + b < p$ and we put $q = \frac{p-(a+b)}{w}$. We also put $\alpha = \sum_{0\leq i}\alpha_i s^i$ and $\beta = \sum_{0\leq i}\beta_i s^i$. With these assumptions, $\ell^{(p)}(res_s(\mathfrak{q})) = 0$ and $\ell^{(p)}(res_{s-xt^w}(\mathfrak{q}')) = x^q \sum_{i+j=q}(b + wj)\alpha_i\beta_j$. In order to compute $res_{s=0}\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q})$, we look at the possible cases:

(i) $\max\{a, b\} < p - (a + b)$
(ii) $p - (a + b) \leq \min\{a, b\}$ and $a \neq b$.
(iii) $p - (a + b) < a$ and $a = b$.
(iv) $p - (a + b) = \max\{a, b\}$ and $a \neq b$
(v) $\min\{a, b\} < p - (a + b) < \max\{a, b\}$.

It is easy to see that these cases are exhaustive under the assumption that $p \geq 5$. Let us look at these cases separately:

(i) In this case,

$$\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}) = -\frac{1}{q}(\frac{x}{s})^q(a\alpha d\beta - b\beta d\alpha)$$

and

$$res_{s=0}\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}) = -\frac{x^q}{q} \sum_{i+j=q} (aj - bi)\alpha_i\beta_j.$$

Since $qw = p - (a+b)$, $-\frac{1}{q}(aj - bi) = -\frac{1}{q}(aj - b(q-j)) = -\frac{a+b}{q}j + b = wj + b$ in characteristic p. Therefore in this case the residue matches with the difference of the $\ell^{(p)}$'s.

(ii) In this case because of the anti-symmetry between the terms we may assume without loss of generality that $a > b$. Then

$$\Omega^{(p)}(\mathfrak{q}'-\mathfrak{q}) = \Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge \underline{e}^{-\sum_{1 \le n < p} \frac{1}{n}(\frac{xt^w}{s})^n}) = \alpha(b\beta d(-\frac{1}{q}\frac{x^q}{s^q}) + w\frac{x^q}{s^q}d\beta) = x^q\alpha(\frac{b\beta ds}{s^{q+1}} + \frac{wd\beta}{s^q}).$$

Hence the residue is

$$x^q \sum_{i+j=q} (b + wj)\alpha_i\beta_j,$$

which again matches exactly to the above difference of $\ell^{(p)}$'s.

(iii) In this case

$$\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}) = \Omega^{(p)}(\underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b} \wedge \underline{e}^{-\sum_{1 \le n < p} \frac{1}{n}(\frac{xt^w}{s})^n}) = -\frac{1}{q}(\frac{x}{s})^q(a\alpha d\beta - b\beta d\alpha)$$

and we proceed exactly as in (i).

(iv) In this case again, without loss of generality, assume that $a < b$. Then we have

$$\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}) = \Omega^{(p)}(\underline{e}^{\beta t^b} \wedge \underline{e}^{-\sum_{1 \le n < p} \frac{1}{n}(\frac{xt^w}{s})^n} \wedge \underline{e}^{\alpha t^a}) = \alpha(b\beta(-\frac{1}{q}d\frac{x^q}{s^q}) + w\frac{x^q}{s^q}d\beta)$$

and we proceed as in (ii).

(v) In this case, again assuming without loss of generality that $a < b$, we proceed exactly as in (iv). This finishes the proof of the lemma. $\square$

We will need the following very elementary observation for the proof of the general proposition on the residues.

Lemma 6.0.2. *Suppose that $s'|_{t^2} = s|_{t^2}$. There exists $u \in k[[s,t]]^\times$ and $x_w \in k$, for $w \ge 2$ such that $s' = us + \sum_{2 \le w} x_w t^w$.*

Proof. By assumption, $s' = s + t^2(\alpha + s\beta)$ with $\alpha \in k[[t]]$ and $\beta \in k[[s,t]]$. If we put $u = 1 + t^2\beta$ and $t^2\alpha = \sum_{2 \le w} x_w t^w$, we obtain the expression we are looking for. $\square$

Lemma 6.0.3. *Suppose that $s'|_{t^2} = s|_{t^2}$ and that $\mathfrak{q}' = s' \wedge \mathfrak{p}$ and $\mathfrak{q} = s \wedge \mathfrak{p}$, where $\mathfrak{p} = \underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b}$, with $0 \le a, b$ and $\alpha, \beta \in k[[s]]$, then we have*

$$(6.0.2) \qquad \ell^{(p)}(res_{s'}(\mathfrak{q}')) - \ell^{(p)}(res_s(\mathfrak{q})) = res_{s=0}\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}).$$

Proof. Let us write $s' = us + \sum_{2 \le w} x_w t^w$ as in Lemma 6.0.2 above. If we let $s'' := us$ and $\mathfrak{q}'' := s'' \wedge \mathfrak{p}$ then $res_{s''}(\mathfrak{q}'') = res_s(\mathfrak{q})$ and since $\Omega^{(p)}(\mathfrak{q}''-\mathfrak{q}) = \Omega^{(p)}(u \wedge \mathfrak{p}) \in \Omega^1_{k[[s]]/k}$, $res_{s=0}\Omega^{(p)}(\mathfrak{q}''-\mathfrak{q}) = 0$.

Let us put $s_1 = s''$ and $s_{i+1} = s_i + x_{i+1}t^{i+1}$ for $1 \le i < p - 1$. If we also let $\mathfrak{q}_i := s_i \wedge \mathfrak{p}$ and apply Lemma 6.0.1 to the pair $\mathfrak{q}_{i+1}$ and $\mathfrak{q}_i$ for all $1 \le i < p - 1$ and take their sum, we obtain

$$\ell^{(p)}(res_{s_{p-1}}(\mathfrak{q}_{p-1})) - \ell^{(p)}(res_{s_1}(\mathfrak{q}_1)) = res_{s=0}\Omega^{(p)}(\mathfrak{q}_{p-1} - \mathfrak{q}_1).$$

Using the previous paragraph, this can be rewritten as:

$$\ell^{(p)}(res_{s_{p-1}}(\mathfrak{q}_{p-1})) - \ell^{(p)}(res_s(\mathfrak{q})) = res_{s=0}\Omega^{(p)}(\mathfrak{q}_{p-1} - \mathfrak{q}).$$

In order finish the proof of the lemma, we need to compare the terms corresponding to s' and s_{p-1}. Note that $s' - s_{p-1} = t^p f(t)$, with $f(t) \in k[[t]]$. This implies that the images of $res_{s'}(s' \wedge \mathfrak{p})$ and $res_{s_{p-1}}(s_{p-1} \wedge \mathfrak{p})$ are equal to each other in $\Lambda^2 k_p^\times$ and hence that $\ell^{(p)}(res_{s'}(\mathfrak{q}')) = \ell^{(p)}(res_{s_{p-1}}(\mathfrak{q}_{p-1}))$. Since $\mathfrak{q}' - \mathfrak{q}_{\mathfrak{p-1}} = (1 + \frac{t^p f(f)}{s_{p-1}}) \wedge \mathfrak{p}$, $\mathfrak{q}' - \mathfrak{q}_{\mathfrak{p-1}}$ has image 0 in $\Lambda^3 k((s))_p$ and hence $\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}_{\mathfrak{p-1}}) = 0$. This completes the proof. $\square$

Lemma 6.0.4. *Suppose that* $a, b \geq 0$, *and* $\alpha, \beta \in k[[s]]$, *then*

$$\ell^{(p)}(res_s(s \wedge \underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b})) = res_{s=0}\Omega^{(p)}(s \wedge \underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b})$$

Proof. Both sides of the expression above are equal to 0 if $a + b \neq p$, or if $a = 0$ or $b = 0$. Suppose that $a + b = p$, and $a, b > 0$. Without loss of generality assume that $a > b$. Then the right hand side is $res_{s=0}(\alpha b \beta \frac{ds}{s}) = b\alpha(0)\beta(0)$. The left hand side is $\ell_p(\underline{e}^{\alpha(0)t^a} \wedge \underline{e}^{\beta(0)t^b}) = b\alpha(0)\beta(0)$. This proves the lemma. $\qquad\square$

We will use the lemmas above to prove the following.

Proposition 6.0.5. *Suppose that* $s|_{t^2} = s'|_{t^2}$ *and* $\mathfrak{q}$ *is* s-*good and* $\mathfrak{q}'$ *is* s'-*good and* $\mathfrak{q}|_{t^2} = \mathfrak{q}'|_{t^2}$. *Then we have*

(6.0.3) $$\ell^{(p)}(res_{s'}(\mathfrak{q}')) - \ell^{(p)}(res_s(\mathfrak{q})) = res_{s=0}\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}).$$

Proof. First, we note that when $\mathfrak{q} \in \Lambda^3 k[[s]]^\times_\infty$ then the same is true for $\mathfrak{q}'$. This shows that the left hand side of the equation is 0 since $res_{s'}(\mathfrak{q}') = res_s(\mathfrak{q}) = 0$. Similarly, the right hand side of the equation is 0 as well since $\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}) \in \Omega^1_{k[[s]]/k}$.

By the multi-linearity and the anti-symmetry of both sides of the equation, we easily reduce to the case where $\mathfrak{q}' = s' \wedge \underline{e}^{\tilde{\alpha}} \wedge \underline{e}^{\tilde{\beta}}$ and $\mathfrak{q} = s \wedge \underline{e}^{\hat{\alpha}} \wedge \underline{e}^{\hat{\beta}}$, where $\tilde{\alpha}, \tilde{\beta}, \hat{\alpha}, \hat{\beta} \in k[[s,t]]$ with $\tilde{\alpha} - \hat{\alpha} \in t^2 k[[s,t]]$ and $\tilde{\beta} - \hat{\beta} \in t^2 k[[s,t]]$. Therefore, in order to prove the proposition, it is enough to prove it in the following two special cases:

(i) $\mathfrak{q}' = s' \wedge \underline{e}^{\tilde{\alpha}} \wedge \underline{e}^{\tilde{\beta}}$ and $\mathfrak{q} = s \wedge \underline{e}^{\tilde{\alpha}} \wedge \underline{e}^{\tilde{\beta}}$ with $\tilde{\alpha}, \tilde{\beta} \in k[[s,t]]$
and
(ii) $\mathfrak{q}' = s \wedge \underline{e}^{\alpha t^a} \wedge \underline{e}^{\beta t^b}$ where $\alpha, \beta \in k[[s]]$ and $a \geq 2$, and $b \geq 0$; and $\mathfrak{q} = 0$.

The first case follows from Lemma 6.0.3. The second case follows from Lemma 6.0.4. Note that here the assumption $a \geq 2$, is not needed in fact. This finishes the proof. $\qquad\square$

Remark. Note that in Lemma 5.2.1, where we show that the residue of $\Omega^{(p)}$ is independent of the coordinates, we only need $w \geq 1$. Therefore, the residues of $\Omega^{(p)}$ are independent of the chosen coordinates when the choice of coordinates agree modulo (t). Recall that, in the characteristic 0 case, which was discussed in [12] and [13], the condition that $w \geq 2$ was essential. This might mistakenly suggest the reader that in characteristic p, one might define the Chow-Kontsevich dilogarithm for a curve over k rather than for a curve over k_2. In fact, this is not true. The argument of comparing the residues of $\Omega^{(p)}$ to the values of $\ell i_2^{(p)}$ does not work if we only consider matching of the parameters only in modulo (t). More precisely, we need $w \geq 2$ in Lemma 6.0.1. The following example shows that this lemma does not hold with $w = 1$.

Let us consider the ring $k[[s,t]]$ and choose the elements

$$\mathfrak{q}' = (s - t) \wedge (1 + s^{p-1}) \wedge (1 + s)$$

and

$$\mathfrak{q} = s \wedge (1 + s^{p-1}) \wedge (1 + s).$$

These two elements are the same modulo (t) but not modulo (t^2). Note that

$$res_{s-t}\mathfrak{q}' = (1 + t^{p-1}) \wedge (1 + t)$$

and since $\ell^{(p)}((1 + t^{p-1}) \wedge (1 + t)) = 1$ we have $\ell^{(p)}(res_{s-t}\mathfrak{q}') = 1$. On the other hand, $res_s(\mathfrak{q}) = 0$. So we have,

$$\ell^{(p)}(res_{s-t}\mathfrak{q}') - \ell^{(p)}(res_s\mathfrak{q}) = 1.$$

On the other hand,

$$\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}) = \Omega^{(p)}((1 - \frac{t}{s}) \wedge (1 + s^{p-1}) \wedge (1 + s)) = 0.$$

Therefore,

$$\ell^{(p)}(res_{s-t}\mathfrak{q}') - \ell^{(p)}(res_s\mathfrak{q}) = 1 \neq 0 = res_{s=0}\Omega^{(p)}(\mathfrak{q}' - \mathfrak{q}).$$

$\qquad\square$

Suppose that $\mathcal{R}$ and $\mathcal{R}'$ are smooth k_p-algebras of relative dimension 1, together with a k_2-isomorphism $\chi : \mathcal{R}/(t^2) \to \mathcal{R}'/(t^2)$ of their reductions modulo (t^2). We identify the spectra of $\mathcal{R}$ and $\mathcal{R}'$ through this isomorphism and assume that c is a closed point and η is the generic point of this spectrum. Let us assume further that $\mathfrak{c}$ and $\mathfrak{c}'$ are smooth liftings of c to $\mathcal{R}$ and $\mathcal{R}'$ and that the reductions of $\mathfrak{c}$ and $\mathfrak{c}'$ modulo (t^2) map to each other under χ. Finally, suppose that $\mathfrak{q} \in \Lambda^3 \mathcal{R}_\eta^\times$ is $\mathfrak{c}$-good and $\mathfrak{q}' \in \Lambda^3(\mathcal{R}'_\eta)^\times$ is $\mathfrak{c}'$-good and that their reductions modulo (t^2) map to each other under χ_η.

Similar to the discussion following (2.2.4), there is a unique isomorphism between the k_p-algebras $k(\mathfrak{c})$ and $k(c)_p$ which is the identity map modulo (t). Using this, we identify $\Lambda^2 k(\mathfrak{c})^\times$ with $\Lambda^2 k(c)_p^\times$. Combining this identification with the map $\ell^{(p)} : \Lambda^2 k(c)_p^\times \to k(c)$, gives us a map

$$\ell^{(p)} : \Lambda^2 k(\mathfrak{c})^\times \to k(c),$$

which we denote by the same symbol.

Corollary 6.0.6. *With the notation as above, we have*

$$res_c \omega^{(p)}(\mathfrak{q}, \mathfrak{q}') = \ell^{(p)}(res_\mathfrak{c}(\mathfrak{q})) - \ell^{(p)}(res_{\mathfrak{c}'}(\mathfrak{q}'))$$

Proof. This is a restatement of Proposition 6.0.5. $\square$

7. An invariant of cycles in characteristic p

We will use the above constructions in order to define the Chow-Kontsevich dilogarithm and in turn use this dilogarithm to construct an infinitesimal invariant of one dimensional cycles in three dimensional space over a field of characteristic p.

7.1. **Proof of Theorem 1.0.1.**
We separate the proof to two parts. In the first part we give the construction of the Chow-Kontsevich dilogarithm. In the second part, we compute this dilogarithm on the projective line.

7.1.1. *Construction of the Chow-Kontsevich dilogarithm.* Suppose that C is a smooth and proper curve over k_2 where k is a field of characteristic $p > 3$. We fix a smooth lifting $\mathfrak{c}$ for each closed point c of C and we let $\mathscr{P}$ be the set of all these smooth liftings as in §2.2.2. We saw in §2.3 that we have a map

$$\rho : \Lambda^3 k(C, \mathscr{P})^\times \to k,$$

which is essentially the same one that was constructed in [12] for the characteristic 0 case.

We will construct another such map

$$\rho_K : \Lambda^3 k(C, \mathscr{P})^\times \to k$$

that is based on the constructions of this paper and has a distinctly characteristic p flavor. We refer to §2.2.2 for the details of the construction of ρ and the notation. We will follow the same notation for the construction of ρ_K.

Suppose that f, g, h are in $k(C, \mathscr{P})^\times$. Denote the triple of functions (f, g, h) by p. Suppose that we fix:

  (i) $\tilde{\mathcal{A}}$, a smooth k_∞-algebra together with an isomorphism $\alpha : \tilde{\mathcal{A}}/(t^2) \xrightarrow{\sim} \mathcal{O}_{C,\eta}$,

  (ii) $\tilde{p}_\eta$, a triple of functions in $\tilde{\mathcal{A}}$, whose reductions modulo (t^2) map to p_η via α,

  (iii) $\widetilde{\mathcal{B}}_c^\circ$, a smooth k_∞-algebra together with an isomorphism $\tilde{\gamma}_c : \widetilde{\mathcal{B}}_c^\circ/(t^2) \xrightarrow{\sim} \hat{\mathcal{O}}_{C,c}$, for each $c \in |C|$,

  (iv) $\tilde{q}_c$, a triple of elements in the localization of $\widetilde{\mathcal{B}}_c^\circ$ at its minimal prime ideal (generic point), which give a good lifting of the image of p_η under the map $\tilde{\gamma}_{c,\eta}^{-1}$, for each $c \in |C|$. Here $\tilde{\gamma}_{c,\eta}^{-1}$ denotes the localization of the inverse of $\tilde{\gamma}_c$ at the generic point η.

Let α_c denote the completion of α at c. The composition $\tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_c$ identifies $\hat{\tilde{\mathcal{A}}}_c/(t^2)$ with $\widetilde{\mathcal{B}}_{c,\eta}^\circ/(t^2)$. Denote the image of the triple of functions $\tilde{p}_\eta$ in the completion $\hat{\tilde{\mathcal{A}}}_c$ at c of $\tilde{\mathcal{A}}$ with the symbol $\tilde{p}_{\eta,c}$. Similarly, denote the image of the triple of functions $\tilde{q}_c$ in the localization $\widetilde{\mathcal{B}}_{c,\eta}^\circ$ of $\widetilde{\mathcal{B}}_c^\circ$ at its generic point, by $\tilde{q}_{c,\eta}$. The reduction of $\tilde{p}_{\eta,c}$ modulo (t^2) is mapped to the reduction of $\tilde{q}_{c,\eta}$ modulo (t^2) by the map $\tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_c$. In precisely this situation, we know that the residue

$res_c\omega^{(p)}(\tilde{p}_{\eta,c}, \tilde{q}_{c,\eta}, \tilde{\gamma}_{c,\eta}^{-1}\circ\alpha_c))$ of $\omega^{(p)}$ at c associated to $\tilde{p}_{\eta,c}$, $\tilde{q}_{c,\eta}$ and the map $\tilde{\gamma}_{c,\eta}^{-1}\circ\alpha_c$ is well-defined by §5.5.

On the other hand, using the notation before (2.2.6), $res_c(\tilde{q}_c) \in \Lambda^2 k(\tilde{\mathfrak{c}})^\times$. Similar to the notation in Corollary 6.0.5, by identifying the k_∞-algebras $k(\tilde{\mathfrak{c}})$ and $k(c)_\infty$ with the unique map which is identity modulo (t), we have a canonical map

$$\ell^{(p)} : \Lambda^2 k(\tilde{\mathfrak{c}})^\times \to k(c).$$

Therefore, we can define $\ell^{(p)}(res_c(\tilde{q}_c)) \in k(c)$.

The value of ρ_K on p is defined by:

$$(7.1.1) \qquad \rho_K(p) := \sum_{c\in|C|} \mathrm{Tr}_k(\ell^{(p)}(res_c(\tilde{q}_c)) + res_c\omega^{(p)}(\tilde{p}_{\eta,c}, \tilde{q}_{c,\eta}, \tilde{\gamma}_{c,\eta}^{-1}\circ\alpha_c)).$$

We need to check that the above sum is finite and is independent of the auxiliary choices.

7.1.2. *The finiteness of (7.1.1).* Let $S \subseteq |C|$ be a finite subset of the set of closed points of C, outside of which all the functions in p and $\tilde{p}_\eta$ are regular. Let $c \in |C| \setminus S$.

Since p is regular at c, so is $\tilde{q}_c$ and hence $res_c(\tilde{q}_c) = 0$. Similarly, since $\tilde{p}_\eta$ is regular at c, $res_c(\tilde{p}_\eta) = 0$. These imply that $\ell^{(p)}(res_c(\tilde{q}_c)) = \ell^{(p)}(res_c(\tilde{p}_\eta)) = 0$. From Corollary 6.0.6, we also deduce that

$$res_c\omega^{(p)}(\tilde{p}_{\eta,c}, \tilde{q}_{c,\eta}, \tilde{\gamma}_{c,\eta}^{-1}\circ\alpha_c) = \ell^{(p)}(res_c(\tilde{p}_\eta)) - \ell^{(p)}(res_c(\tilde{q}_c)) = 0.$$

This implies that the sum in (7.1.1) is in fact over S and hence is finite.

7.1.3. *Independence of (7.1.1) from the choices.* Fix a $c \in |C|$. Let $\widehat{\mathcal{B}}_c'^{\circ}$, be another smooth k_∞-algebra together with an isomorphism $\tilde{\gamma}_c' : \widehat{\mathcal{B}}_c'^{\circ}/(t^2) \xrightarrow{\sim} \hat{\mathcal{O}}_{C,c}$ and $\tilde{q}_c'$, a triple of elements in the localization of $\widehat{\mathcal{B}}_c'^{\circ}$ at its minimal prime ideal which gives a good lifting of the image of p_η under the inverse of the map $\tilde{\gamma}_{c,\eta}'$.

The difference of

$$\ell^{(p)}(res_c(\tilde{q}_c)) + res_c\omega^{(p)}(\tilde{p}_{\eta,c}, \tilde{q}_{c,\eta}, \tilde{\gamma}_{c,\eta}^{-1}\circ\alpha_c)$$

and

$$\ell^{(p)}(res_c(\tilde{q}_c')) + res_c\omega^{(p)}(\tilde{p}_{\eta,c}, \tilde{q}_{c,\eta}', (\tilde{\gamma}_{c,\eta}')^{-1}\circ\alpha_c)$$

is equal to

$$\ell^{(p)}(res_c(\tilde{q}_c)) - \ell^{(p)}(res_c(\tilde{q}_c')) - res_c\omega^{(p)}(\tilde{q}_{c,\eta}, \tilde{q}_{c,\eta}', (\tilde{\gamma}_{c,\eta}')^{-1}\circ\tilde{\gamma}_{c,\eta})$$

by Lemma 5.5.3. On the other hand, the last expression is 0 by Corollary 6.0.6. This proves independence from the local choices.

Next we prove independence from the global choice. Let $\tilde{\mathcal{A}}'$ be a smooth k_∞-algebra together with an isomorphism $\alpha' : \tilde{\mathcal{A}}'/(t^2) \xrightarrow{\sim} \mathcal{O}_{C,\eta}$ and $\tilde{p}_\eta'$, be a triple of functions in $\tilde{\mathcal{A}}'$, whose reductions modulo (t^2) map to p_η via α'. Similar to above, the difference of

$$\ell^{(p)}(res_c(\tilde{q}_c)) + res_c\omega^{(p)}(\tilde{p}_{\eta,c}, \tilde{q}_{c,\eta}, \tilde{\gamma}_{c,\eta}^{-1}\circ\alpha_c)$$

and

$$\ell^{(p)}(res_c(\tilde{q}_c)) + res_c\omega^{(p)}(\tilde{p}_{\eta,c}', \tilde{q}_{c,\eta}, \tilde{\gamma}_{c,\eta}^{-1}\circ\alpha_c')$$

is equal to

$$res_c\omega^{(p)}(\tilde{p}_{\eta,c}, \tilde{p}_{\eta,c}', (\alpha_c')^{-1}\circ\alpha_c)$$

by Lemma 5.5.3. This implies that the difference between the expressions (7.1.1) corresponding to $\tilde{p}_\eta$ and $\tilde{p}_\eta'$ is equal to

$$\sum_{c\in|C|} \mathrm{Tr}_k(res_c\omega^{(p)}(\tilde{p}_{\eta,c}, \tilde{p}_{\eta,c}', (\alpha_c')^{-1}\circ\alpha_c)).$$

Let us choose an isomorphism $\beta : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}'$ whose reduction modulo (t) is the same as the reduction of $(\alpha')^{-1}\circ\alpha$, and $\varphi : k(C)_\infty \to \tilde{\mathcal{A}}_\eta$ is any k-algebra isomorphism. Let $\tilde{q}$ denote the inverse image of $\tilde{p}_\eta$ under φ and $\tilde{q}'$ denote the inverse image of $\tilde{p}_\eta'$ under $\beta_\eta\circ\varphi$. Then $\tilde{q}$ and $\tilde{q}'$ are triples of functions in $k(C)_\infty$ with the same reduction modulo (t).

By its definition, for any $c \in |C|$, $res_c \omega^{(p)}(\tilde{p}_{\eta,c}, \tilde{p}'_{\eta,c}, (\alpha'_c)^{-1} \circ \alpha_c)$ is equal to the residue of

$$\Omega^{(p)}(\tilde{q} - \tilde{q}') \in \Omega^1_{k(C)/k}$$

at c. Since the sum of the residues of a 1-form is 0, this finishes the proof of the independence.

7.1.4. *Computation of the Chow-Kontsevich dilogrithm on the projective line.* Using functoriality with respect to the map that sends z to $\frac{z-\beta}{\alpha-\beta}$, we will assume without loss of generality that $\alpha = 1$, $\beta = 0$ and $\gamma = s + as(1-s) \cdot t$. Therefore we need to compute $\rho_K((1-z) \wedge z \wedge (z - (s + as(1-s) \cdot t)))$. In order to do this computation, we may choose the global lifting $\mathbb{P}^1_{k_\infty}$ and the triple of functions $\tilde{p} := (1 - z) \wedge z \wedge (z - (s + as(1 - s) \cdot t))$ as a global lift of the functions. Let us put $\tilde{s} := s + as(1-s) \cdot t$. The only singularities of $\tilde{p}$ are at $0, 1, \infty$ and $\tilde{s}$. Since $res_\delta \tilde{p} = 0$ for $\delta \in \{0, 1, \infty\}$, and $res_{\tilde{s}} = (1 - \tilde{s}) \wedge \tilde{s}$, we find that

$$\rho_K((1 - z) \wedge z \wedge (z - (s + as(1 - s) \cdot t))) = \ell^{(p)}((1 - \tilde{s}) \wedge \tilde{s}) = \ell i_2^{(p)}(\tilde{s}) = a^p \cdot \mathcal{L}_1(s).$$

7.2. **An infinitesimal invariant of cycles in characteristic p.** We constructed an invariant of codimension two cycles in the three dimensional space over a field of characteristic 0 in [12, §4] which we denoted by ρ_f. In characteristic $p \geq 5$, this construction carries without any modification and we denote this regulator by ρ. On the other hand, in characteristic p, there is another regulator which is based on the Chow-Kontsevich dilogarithm and which we will denote by ρ_K. This new regulator has no characteristic 0 analog.

Since the definitions and the proofs are exactly analogous to those in characteristic 0, we will omit most of the details and refer the reader to [12, §4]. We start by recalling the definitions of Bloch's higher Chow groups for a smooth scheme over a field and its version for the truncated polynomial ring over a field from [12, §4].

7.2.1. *Bloch's higher Chow groups.* Let $\Box_k := \mathbb{P}^1_k \setminus \{1\}$ and $\Box^n_k$ the n-fold product of $\Box_k$ with itself over k, with the coordinate functions $y_1, \cdots, y_n$. For a smooth k-scheme X, we let $\Box^n_X := X \times_k \Box^n_k$. A codimension one face of $\Box^n_X$ is a divisor F_i^a of the form $y_i = a$, for $1 \leq i \leq n$, and $a \in \{0, \infty\}$. A face of $\Box^n_X$ is either the whole scheme $\Box^n_X$ or an arbitrary intersection of codimension one faces.

Let $\underline{z}^q(X, n)$ be the free abelian group on the set of codimension q, integral, closed subschemes $Z \subseteq \Box^n_X$ which are *admissable*, i.e. which intersect each face properly on $\Box^n_X$. For each codimension one face F_i^a, and irreducible $Z \in \underline{z}^q(X, n)$, we let $\partial_i^a(Z)$ be the cycle associated to the scheme $Z \cap F_i^a$. We let $\partial := \sum_{i=1}^n (-1)^n (\partial_i^\infty - \partial_i^0)$ on $\underline{z}^q(X, n)$.

Let $\underline{z}^q(X, n)_{\text{degn}}$ denote the subgroup of degenerate cycles and $z^q(X, \cdot) := \underline{z}^q(X, \cdot)/\underline{z}^q(X, \cdot)_{\text{degn}}$ the corresponding non-degenerate complex. The complex $(z^q(X, \cdot), \partial)$ is called the *higher Chow complex* of X and its homology $\text{CH}^q(X, n) := \text{H}_n(z^q(X, \cdot))$ is the higher Chow group of X.

7.2.2. *Cycles over k_∞.* Let $\overline{\Box}_k := \mathbb{P}^1_k$, $\overline{\Box}^n_k$, the n-fold product of $\overline{\Box}_k$ with itself over k, and $\overline{\Box}^n_{k_\infty} := \overline{\Box}^n_k \times_k k_\infty$. We define a subcomplex $\underline{z}^q_f(k_\infty, \cdot) \subseteq \underline{z}^q(k_\infty, \cdot)$, as the subgroup generated by integral, closed subschemes $Z \subseteq \Box^n_{k_\infty}$ which are admissible and have the property that $\overline{Z}$ intersects each $s \times \overline{F}$ properly on $\overline{\Box}^n_{k_\infty}$, for every face F of $\Box^n_k$. Here s denotes the closed point of $\text{Spec}\, k_\infty$. We refer to such cycles as cycles having finite reduction. Modding out by degenerate cycles, we have a complex $z^q_f(k_\infty, \cdot)$.

7.2.3. *Definition of the invariant.* Let η denote the generic point of $\text{Spec}\, k_\infty$. An irreducible cycle q in $\underline{z}^2_f(k_\infty, 2)$ is determined by its generic point q_η of $\Box^2_\eta$, such that its closure $\overline{q}$ in $\overline{\Box}^2_{k_\infty}$ does not meet $(\{0, \infty\} \times \overline{\Box}_{k_\infty}) \cup (\overline{\Box}_{k_\infty} \times \{0, \infty\})$. Let $\tilde{q}$ denote the normalisation of $\overline{q}$ and $|\tilde{q}_s|$ denote the underlying set of the closed fiber $\tilde{q} \times_{k_\infty} s$ of $\tilde{q}$. Let $\pi : \tilde{q} \to (\overline{\Box}_{k_\infty} \setminus \{0, \infty\})^2$ denote the composition of the normalization map from $\tilde{q}$ to $\overline{q}$ and the inclusion of $\overline{q}$ to $(\overline{\Box}_{k_\infty} \setminus \{0, \infty\})^2$.

For $r \in |\tilde{q}_s|$, let $k(r)$ denote the residue field of r. We have a surjection $\hat{\mathcal{O}}_{\tilde{q}, r} \to k(r)$. Since $k(r)/k$ is finite étale there is a unique splitting $k(r) \to \hat{\mathcal{O}}_{\tilde{q}, r}$ of the above surjection, and a unique isomorphism $k(r)_\infty \xrightarrow{\sim} \hat{\mathcal{O}}_{\tilde{q}, r}$ of k_∞-algebras which extend this splitting.

We let π_r denote the composition of the isomorphism $\operatorname{Spec} k(r)_\infty \xrightarrow{\sim} \operatorname{Spec} \hat{\mathcal{O}}_{\tilde{q},r}$, the natural map $\operatorname{Spec} \hat{\mathcal{O}}_{\tilde{q},r} \to \tilde{q}$ and the map π. If y is an invertible function on $(\overline{\square}_{k_\infty} \setminus \{0,\infty\})^2$, we let

$$\ell_{r,i}(y) := \ell_i(\pi_r^*(y)),$$

for $1 \le i < p$. Similarly, we let

$$\ell(r) := (\ell_{r,2} \wedge \ell_{r,1})(y_1 \wedge y_2)$$

and

$$\ell^{(p)}(r) := \frac{1}{2} \sum_{1 \le i < p} i(\ell_{r,p-i} \wedge \ell_{r,i})(y_1 \wedge y_2),$$

where y_1 and y_2 are the coordinate functions on $(\overline{\square}_{k_\infty} \setminus \{0,\infty\})^2$.

We define the values of ℓ and $\ell^{(p)}$ on q by:

(7.2.1) $$\ell(q) := \sum_{r \in |\tilde{q}_s|} \operatorname{Tr}_k(\ell(r))$$

and

(7.2.2) $$\ell^{(p)}(q) := \sum_{r \in |\tilde{q}_s|} \operatorname{Tr}_k(\ell^{(p)}(r)).$$

Definition 7.2.1. We define the regulators ρ and $\rho^{(p)}$ as maps from $\underline{z}_f^2(k_\infty, 3)$ to k as the compositions of ℓ and $\ell^{(p)}$ with the boundary map:

$$\rho := \ell \circ \partial \qquad \text{and} \qquad \rho_K := \ell^{(p)} \circ \partial.$$

Exactly as in [12], we see that ρ and ρ_K vanish on boundaries. In other words, the compositions $\rho \circ \partial$ and $\rho_K \circ \partial$ from $\underline{z}_f^2(k_\infty, 4)$ to k are both 0.

7.2.4. *Modulus property.* The most important property of the regulator maps ρ and ρ_K states, in essence, that these regulators depend only on the reduction of the cycle modulo (t^2). In order to state this property precisely, we will need the following definition.

Definition 7.2.2. Suppose that $Z_i \in \underline{z}_f^2(k_\infty, 3)$, for $i = 1, 2$, are irreducible cycles. We say that Z_1 and Z_2 are *equivalent modulo t^m* if the following condition, which we denote by (M_m), holds:
   (i) the closure $\overline{Z}_i$ of Z_i is smooth over k_∞ and $(\overline{Z}_i)_s \cup (\cup_{j,a} \partial_j^a Z_i|)$ is a strict normal crossings divisor on $\overline{Z}_i$, for $i = 1, 2$
   (ii) $\overline{Z}_1|_{t^m} = \overline{Z}_2|_{t^m}$.

The main result of this section is the following:

Theorem 7.2.3. *Suppose that $Z_i \in \underline{z}_f^2(k_\infty, 3)$, for $i = 1, 2$, are two irreducible cycles which are equivalent modulo t^2. Then we have*

$$\rho(Z_1) = \rho(Z_2)$$

and

$$\rho_K(Z_1) = \rho_K(Z_2).$$

Proof. The proofs of the statements for ρ and ρ_K are essentially the same. Therefore, we will only expound the one for ρ_K.

Suppose that Z_1 and Z_2 are as in the statement. Let us put $\overline{Z}$ to be the common reduction of $\overline{Z}_i$ modulo (t^2). Then $\overline{Z}$ is a smooth and projective curve over k_2 and $\overline{Z}_i$ are its two different liftings. We will denote the restriction of the coordinate function y_j to $\overline{Z}$ by $y_{j,\overline{Z}}$, similarly its restriction to $\overline{Z}_i$ by $y_{j,\overline{Z}_i}$.

Let us compute $\rho_K(y_{1,\overline{Z}} \wedge y_{2,\overline{Z}} \wedge y_{3,\overline{Z}})$ by using the lifting $\overline{Z}_i$. Note that we proved in the main theorem that the value of $\rho_K(y_{1,\overline{Z}} \wedge y_{2,\overline{Z}} \wedge y_{3,\overline{Z}})$ is independent of the choice of the lifting.

Since $\overline{Z}_i$ and the functions $y_{j,\overline{Z}_i}$ are global good liftings, we can choose $y_{1,\overline{Z}_i} \wedge y_{2,\overline{Z}_i} \wedge y_{3,\overline{Z}_i}$ as both $\tilde{p}_\eta$ and $\tilde{q}_c$ for each $z \in |\overline{Z}|$ as in the construction in §7.1.1. This would make the defect term

$$res_z \omega^{(p)}(\tilde{p}_{\eta,z}, \tilde{q}_{z,\eta}, \tilde{\gamma}_{z,\eta}^{-1} \circ \alpha_z)$$

equal to 0 in (7.1.1) and the expression reduces to

$$\rho_K(y_{1,\overline{Z}} \wedge y_{2,\overline{Z}} \wedge y_{3,\overline{Z}}) = \sum_{z \in |\overline{Z}|} \mathrm{Tr}_k(\ell^{(p)}(res_z(y_{1,\overline{Z}_i} \wedge y_{2,\overline{Z}_i} \wedge y_{3,\overline{Z}_i}))) = (\ell^{(p)} \circ \partial)(Z_i) = \rho_K(Z_i).$$

Since the left hand side does not depend on the choice of Z_i, we obtain $\rho_K(Z_1) = \rho_K(Z_2)$. $\qquad \square$

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