INFINITESIMAL DILOGARITHM ON CURVES OVER TRUNCATED POLYNOMIAL RINGS

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Abstract. We construct infinitesimal invariants of thickened one dimensional cycles in three dimensional space, which are the simplest cycles that are not in the Milnor range. This generalizes Park's work on the regulators of additive cycles. The construction also allows us to prove the infinitesimal version of the strong reciprocity conjecture for thickenings of all orders. Classical analogs of our invariants are based on the dilogarithm function and our invariant could be seen as their infinitesimal version. Despite this analogy, the infinitesimal version cannot be obtained from their classical counterparts through a limiting process.

1. Introduction

1.1. **Statement of the main technical result.** For a scheme X, one expects an abelian category $\mathcal{M}_X$ of mixed motivic $\mathbb{Q}$-sheaves on X, such that the extensions groups $\mathrm{H}^i_{\mathcal{M}}(X, \mathbb{Q}(n)) := \mathrm{Ext}^i_{\mathcal{M}_X}(\mathbb{Q}(0), \mathbb{Q}(n))$ of the Tate sheaves are computed in terms of the K-groups as $K_{2n-i}(X)^{(n)}_{\mathbb{Q}}$. We emphasize that we do not assume that X is smooth over a field or even reduced. At present, such a category has not been constructed. When X is a smooth and projective curve over a base scheme S, which in our case will be the spectrum of an artin ring, the conjectural Leray-Serre spectral sequence would give a map:

$$K_3(X)^{(3)}_{\mathbb{Q}} = \mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3)) \to \mathrm{H}^1_{\mathcal{M}}(S, \mathbb{Q}(2)) = K_3(S)^{(2)}_{\mathbb{Q}}.$$

In certain cases, there are regulator maps from $K_3(S)^{(2)}_{\mathbb{Q}}$ to an abelian group A. The composition with the above map would induce a map from $K_3(X)^{(3)}_{\mathbb{Q}}$ to A. In case $S = \mathrm{Spec}\, k[t]/(t^m)$, such a map $K_3(S)^{(2)}_{\mathbb{Q}} \to \oplus_{m<r<2m} k$ was constructed in [10]. One of our aims in this paper is to give an analog of the induced map

$$\mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3)) = K_3(X)^{(3)}_{\mathbb{Q}} \to \oplus_{m<r<2m} k,$$

which does not depend on the conjectural category of motives. This map is an infinitesimal analog of a real analytic regulator as we will describe in §2.2. This makes this paper a continuation of our project started in [21] and followed up in [20], which aim to give infinitesimal analogs of real analytic regulators.

First, let us state the main technical result on which all the applications are based. Let k be a field of characteristic 0, $k_m := k[t]/(t^m)$, for $m \geq 2$, and C/k_m be a smooth and projective curve. We denote the underlying reduced scheme of C by $\underline{C}$. We will need a variant of the Bloch complex ([6, §1.8, §1.9], §2.1). If X/k is a smooth and projective curve, then the part of the classical Bloch complex relevant for us is:

$$B_2(k(X)) \otimes k(X)^{\times} \to \oplus_{x \in |X|} B_2(k(x)) \oplus \Lambda^3 k(X)^{\times} \to \oplus_{x \in |X|} \Lambda^2 k(x)^{\times}. \tag{1.1.1}$$

Here the summations are over the set closed points $|X|$ of X and B_2 denotes the Bloch group (§2.1.1).

In order to define a variant of the above complex for C, we first need to make a choice of smooth liftings. By a smooth lifting $\mathfrak{c}$ of a closed point $c \in |C|$, we mean a closed subscheme $\mathfrak{c}$ of C, which is supported on c and is smooth over k_m (cf. §7). For each point $c \in |C|$, fix once and for all a smooth lifting $\mathfrak{c}$ and let $\mathscr{P}$ denote the set of all of these liftings. Let η be the generic point of C, for a function $f \in \mathcal{O}^{\times}_{C,\eta}$, and $\mathfrak{c} \in \mathscr{P}$, we define a notion of f being *good with respect to*

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$\mathfrak{c}$ or equivalently of being $\mathfrak{c}$-*good* in §7. We then define the sheaf $(\mathcal{O}_C, \underline{\mathscr{P}})^\times$ in §8.1, by requiring that its sections on an open set U to be those $f \in \mathcal{O}_{C,\eta}^\times$, which are $\mathfrak{c}$-good for all $\mathfrak{c} \in \mathscr{P}$ with $c := |\mathfrak{c}| \in U$. We similarly define a sheaf $B_2(\mathcal{O}_C, \underline{\mathscr{P}})$ in §8.1, which is a generalization of the Bloch group but which also encodes the notion of goodness with respect to elements of $\mathscr{P}$. This gives us a complex $\mathscr{C}(C, \mathscr{P})$ of sheaves on C which are concentrated in degrees 2 and 3:

$$B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^\times \to \oplus_{\mathfrak{c} \in \mathscr{P}} i_{\mathfrak{c}*}(B_2(k(\mathfrak{c}))) \oplus \Lambda^3(\mathcal{O}_C, \underline{\mathscr{P}})^\times. \tag{1.1.2}$$

Here $k(\mathfrak{c})$ denotes the artin ring which is the ring of regular functions on the affine scheme $\mathfrak{c}$, and $i_{\mathfrak{c}}$ denotes the imbedding from $\mathfrak{c}$ to C. Since we fixed a single lifting $\mathfrak{c} \in \mathscr{P}$ for each point c in $|C|$, the sum above can also be thought of as a sum over $|C|$. The main technical result is the following construction of infinitesimal Chow dilogarithms:

Theorem 8.1.1. *Let k be a field of characteristic 0, C be a smooth and projective curve over $k_m := k[t]/(t^m)$, with $m \geq 2$ and $\mathscr{P}$ be a choice of a smooth lifting for each closed point of C. For each $m < r < 2m$, there is an infinitesimal regulator:*

$$\rho_{m,r} : \mathrm{H}^3(\mathscr{C}(C, \mathscr{P})) \to k. \tag{1.1.3}$$

Specializing to the case when C is the projective line $\mathbb{P}^1_{k_m}$, with coordinate function z, we fix an $a \in k_m^\times$ such that $1 - a \in k_m^\times$. If we choose $\mathscr{P}$ such that that z, $1 - z$ and $z - a$ are all good with respect to $\mathscr{P}$, then $(1 - z) \wedge z \wedge (z - a) \in \Gamma(\Lambda^3(\mathcal{O}_{\mathbb{P}^1}, \underline{\mathscr{P}})^\times)$ and

$$\rho_{m,r}((1 - z) \wedge z \wedge (z - a)) = \ell i_{m,r}([a]),$$

where $\ell i_{m,r} : B_2(k_m) \to k$ is the additive dilogarithm defined in [18] *(cf. §3).*
The notation of the theorem in the main body of the paper is slightly different but equivalent.

This generalizes the construction in [21] in two different ways: we sheafify the previous construction and we construct the regulator for any $m < r < 2m$, rather than only for $m = 2$. More precisely, if we let $k(C, \underline{\mathscr{P}})^\times$ denote the set of global sections $\Gamma(C, (\mathcal{O}_C, \underline{\mathscr{P}})^\times)$ of $(\mathcal{O}_C, \underline{\mathscr{P}})^\times$, the construction of [21] only gives a map from $\Lambda^3 k(C, \underline{\mathscr{P}})^\times$ and only in the case when $m = 2$ and $r = 3$. We will sketch the main idea of the construction in the section below, but let us mention here that the construction of a map $\rho_{m,m+1}$ from $\Lambda^3 k(C, \underline{\mathscr{P}})^\times$ to k can be done by the methods of [21]. On the other hand, the construction of $\rho_{m,r}$ for $m+1 < r < 2m$ requires the new methods that we introduce in this paper.

1.2. **Applications.** As we described above, specializing to triples of functions gives us the infinitesimal Chow dilogarithm:

$$\rho_{m,r} : \Lambda^3 k(C, \underline{\mathscr{P}})^\times \to k. \tag{1.2.1}$$

which we will denote by the same symbol.

1.2.1. *Infinitesimal strong reciprocity conjecture.* The first application of this construction will be to an infinitesimal analog of the strong reciprocity conjecture of Goncharov [7]. If X/k is a smooth and projective curve over an algebraically closed field k, the Suslin reciprocity theorem states that the sum of the residue maps

$$\sum_{x \in |X|} res_x : K_3^M(k(X)) \to K_2^M(k) \tag{1.2.2}$$

at all the closed points of X is equal to 0. In [7], Goncharov conjectures that the map of complexes:

$$\begin{array}{ccc} B_2(k(X)) \otimes k(X)^\times & \longrightarrow & \Lambda^3 k(X)^\times \\ \downarrow & & \downarrow \\ B_2(k) & \longrightarrow & \Lambda^2 k^\times \end{array}$$

obtained from (1.1.1) by taking sums of the maps $B_2(k(x)) \to B_2(k)$ and $\Lambda^2 k(x)^\times \to \Lambda^2 k^\times$, is homotopic to 0. More precisely, he conjectures that there is a canonical map $h : \Lambda^3 k(X)^\times \to B_2(k)$ which makes the diagram

$$\begin{array}{ccc} B_2(k(X)) \otimes k(X)^\times & \longrightarrow & \Lambda^3 k(X)^\times \\ \downarrow & \overset{h}{\swarrow} & \downarrow \\ B_2(k) & \longrightarrow & \Lambda^2 k^\times \end{array}$$

commute and has the property that $h(\lambda \wedge f \wedge g) = 0$, if $\lambda \in k^\times$ and $f,\, g \in k(X)^\times$. Note that this is a stronger version of the Suslin reciprocity theorem since the cokernel of the horizontal maps in the diagram above are $K_3^M(k(X))$ and $K_2^M(k)$. This original version of the conjecture is proved by Rudenko [16], by using homotopy invariance.

We prove an infinitesimal version of this conjecture using the infinitesimal Chow dilogarithm above and the determination of the structure of the Bloch group over k_m which was done in [18]. Our method is entirely different from Rudenko's, since homotopy invariance is no longer true in the infinitesimal world. Let $B_2(k(C, \underline{\mathscr{P}}))$ denote the set of global sections of $B_2(\mathcal{O}_C, \underline{\mathscr{P}})$, then the infinitesimal version of the strong reciprocity conjecture states:

Theorem 9.1.1. *There is a map $h : \Lambda^3 k(C, \underline{\mathscr{P}})^\times \to B_2(k_m)$, which makes the diagram*

$$\begin{array}{ccc} B_2(k(C, \underline{\mathscr{P}})) \otimes k(C, \underline{\mathscr{P}})^\times & \longrightarrow & \Lambda^3 k(C, \underline{\mathscr{P}})^\times \\ \downarrow & \overset{h}{\swarrow} & \downarrow \\ B_2(k_m) & \longrightarrow & \Lambda^2 k_m^\times \end{array}$$

commute and has the property that $h(k_m^\times \wedge \Lambda^2 k(C, \underline{\mathscr{P}})^\times) = 0$.

1.2.2. *Application to algebraic cycles.* As another application of the infinitesimal Chow dilogarithm, we construct invariants of higher algebraic cycles up to rational equivalence. In principle the group of algebraic cycles that we are interested in should be denoted by $\mathrm{CH}^2(k_m, 3)$. However, since k_m is far from being smooth over k, such a group of cycles which can be expressed in terms of K-theory is not defined.

One way overcoming this problem is to use the additive Chow groups of Bloch and Esnault [4]. Additive Chow groups were defined in order to give a cycle theoretic interpretation of the motivic cohomology groups of k_m. One can think of additive Chow cycles as those cycles which are very close to the 0 cycle, the closeness to 0 being defined via the modulus (t^m). A regulator on this group was defined by Park in [15] for $r = m + 1$.

We think that additive Chow groups tell only part of the story when we try to understand higher cycles on k_m. For this reason we define a somewhat bigger class of higher cycles over k_m. We do this by defining a group $\underline{z}_f^2(k_\infty, 3)$ of codimension 2 cycles on $\mathbb{A}^3_{k_\infty}$, where $k_\infty := k[[t]]$. The main theorem is then a reciprocity theorem:

Theorem 9.4.2. *For $m < r < 2m$, we define a regulator $\rho_{m,r} : \underline{z}_f^2(k_\infty, 3) \to k$. If Z_a, for $a(1-a) \in k_\infty^\times$ is the dilogarithmic cycle given by the parametric equation $(1-z, z, z-a)$ then*

$$\rho_{m,r}(Z_a) = \ell i_{m,r}([a]).$$

If $Z_i \in \underline{z}_f^2(k_\infty, 3)$, for $i = 1, 2$, satisfy the condition (M_m), then they have the same infinitesimal regulator value:

$$\rho_{m,r}(Z_1) = \rho_{m,r}(Z_2).$$

This essentially states that if two cycles are the same modulo (t^m) then they have the same value under the regulator. Note the similarity of this to the definition of de Rham cohomology on singular schemes by first imbedding them in a smooth scheme. The precise definition of $\underline{z}_f^2(k_\infty, 3)$ and the condition (M_m) can be found in §9.4. After the category of motives over non-reduced rings is constructed, we expect these invariants to induce the regulators in this category.

1.3. **Main ideas behind the construction.** In this section, we will try to illustrate the ideas behind the construction in Theorem 8.1.1. For each $2 \leq m < r < 2m$, we will construct a regulator whose source is the degree 3 cohomology of the complex of sheaves:

$$B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^{\times} \to \oplus_{\mathfrak{c} \in \mathscr{P}} i_{\mathfrak{c}*}(B_2(k(\mathfrak{c}))) \oplus \Lambda^3(\mathcal{O}_C, \underline{\mathscr{P}})^{\times}$$

concentrated in the degrees $[2, 3]$.

Suppose that we are given a Zariski open cover $\{U_i\}_{i \in I}$ of $\underline{C}$ and a corresponding cocyle γ, given by the following data: $\gamma_i \in \Lambda^3(\mathcal{O}_C, \underline{\mathscr{P}})^{\times}(U_i)$; $\varepsilon_{i,c} \in B_2(k(\mathfrak{c}))$ for every $c \in U_i$; and $\beta_{ij} \in (B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^{\times})(U_{ij})$. We will define $\rho_{m,r}(\gamma) \in k$, by first making many choices and then showing that the construction is independent of all the choices.

(i) Let $\tilde{\mathcal{A}}_\eta$ be a lifting of $\mathcal{O}_{C,\eta}$ to a smooth k_∞-algebra and for every $c \in |C|$, let $\tilde{\mathcal{A}}_c$ be a lifting of the completion $\hat{\mathcal{O}}_{C,c}$ of the local ring of C at c, to a smooth k_∞-algebra, together with a smooth lifting $\tilde{\mathfrak{c}}$ of $\mathfrak{c}$.

(ii) Let an $i \in I$ be arbitrary and for each c choose a $j_c \in I$ such that $c \in U_{j_c}$

(iii) Choose an arbitrary lifting $\tilde{\gamma}_{i\eta} \in \Lambda^3 \tilde{\mathcal{A}}_\eta^{\times}$ of the germ $\gamma_{i\eta} \in \Lambda^3 \mathcal{O}_{C,\eta}^{\times}$

(iii) Choose a good lifting $\tilde{\gamma}_{j_c} \in \Lambda^3(\tilde{\mathcal{A}}_c, \tilde{\mathfrak{c}})^{\times}$ of the image $\hat{\gamma}_{j_c,c}$ of γ_{j_c} in $\Lambda^3(\hat{\mathcal{O}}_{C,c}, \mathfrak{c})^{\times}$, for every $c \in |C|$,

(iv) Choose an arbitrary lifting $\tilde{\beta}_{j_c i,\eta} \in B_2(\tilde{\mathcal{A}}_\eta) \otimes \tilde{\mathcal{A}}_\eta$ of the image $\beta_{j_c i,\eta} \in B_2(\mathcal{O}_{C,\eta}) \otimes \mathcal{O}_{C,\eta}^{\times}$ of $\beta_{j_c i}$, for every $c \in |C|$.

We then define the value of the regulator $\rho_{m,r}$ on the above element by the expression

$$\rho_{m,r}(\gamma) := \sum_{c \in |C|} \mathrm{Tr}_k\big(\ell_{m,r}(res_{\tilde{\mathfrak{c}}} \tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + res_c \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i,\eta}), \tilde{\gamma}_{j_c})\big). \tag{1.3.1}$$

We continue with the description of this expression.

The starting point for the above definition is our construction of the additive dilogarithm in [18]. For a regular local $\mathbb{Q}$-algebra R, letting $R_m := R[t]/(t^m)$, for every $2 \leq m < r < 2m$, we have an additive dilogarithm map $\ell i_{m,r} : B_2(R_m) \to R$ that satisfies all the analogous properties of the Bloch-Wigner dilogarithm function. Most importantly, the direct sums of these maps over all the possible r's give an isomorphism between the infinitesimal part of the K-group $K_3(R_m)^{(2)}_{\mathbb{Q}}$ and $\oplus_{m<r<2m} R$. We explain this in detail in §3 and give explicit formulas for these functions $\ell i_{m,r}$. The function $\ell i_{m,r}$ can also be described in terms of the differential δ in the Bloch complex of $B_2(R_\infty)$, with $R_\infty := R[[t]]$, by the following commutative diagram

$$\begin{array}{ccc} B_2(R_\infty) & \xrightarrow{\delta} & \Lambda^2 R_\infty^{\times} \\ \downarrow & & \downarrow \ell_{m,r} \\ B_2(R_m) & \xrightarrow{\ell i_{m,r}} & R, \end{array}$$

where $\ell_{m,r}$ is given explicitly in Definition 3.0.2 below.

We can then describe the first two terms in (1.3.1) as follows. For a connected, étale k_m-algebra (resp. k_∞-algebra) A, there is a canonical isomorphism $A \simeq k'_m$ (resp. $A \simeq k'_\infty$). Using this isomorphism for $k(\mathfrak{c})$, we get a canonical identification $B_2(k(\mathfrak{c})) = B_2(k(c)_m)$. Therefore, $\ell i_{m,r}(\varepsilon_{j,c}) \in k(c)$ is unambiguously defined using the map $\ell i_{m,r} : B_2(k(c)_m) \to k(c)$. Since the element $\tilde{\gamma}_{j_c} \in \Lambda^3(\tilde{\mathcal{A}}_c, \tilde{\mathfrak{c}})^{\times}$ is assumed to be $\tilde{\mathfrak{c}}$-good, the residue $res_{\tilde{\mathfrak{c}}} \tilde{\gamma}_{j_c}$ is defined as an element of $\Lambda^2 k(\tilde{\mathfrak{c}})^{\times}$ in the beginning of §7. Using the identification $\Lambda^2 k(\tilde{\mathfrak{c}})^{\times} = \Lambda^2 k(c)_\infty^{\times}$ and the map $\ell_{m,r} : \Lambda^2 k(c)_\infty^{\times} \to k(c)$, we define the element $\ell_{m,r}(res_{\tilde{\mathfrak{c}}} \tilde{\gamma}_{j_c}) \in k(c)$.

Defining the last term $res_c \omega_{m,r}$ and proving its properties will constitute a large proportion of the paper. If R is smooth of relative dimension 1 over k, we construct a map $\omega_{m,r}$: $\Lambda^3(R_r, (t^m))^{\times} \to \Omega^1_{R/k}$. Here $(R_r, (t^m))^{\times}$ denotes $\{(a, b) | a, b \in R_r^{\times},\ ab^{-1} \in 1 + (t^m)\}$. Since we do not fix a lifting of our curve in the construction of $\rho_{m,r}$, defining $\omega_{m,r}$ on this group is not enough. More precisely, we need to extend $\omega_{m,r}$ to the following context. Suppose that $\mathcal{R}$ and $\mathcal{R}'$ are smooth of relative dimension 1 over k_r together with a fixed isomorphism:

$$\chi : \mathcal{R}/(t^m) \to \mathcal{R}'/(t^m),$$

of k_m-algebras between their reductions modulo (t^m). Let

$$(\mathcal{R}, \mathcal{R}', \chi)^\times := \{(a, b)|a \in \mathcal{R}^\times,\ b \in \mathcal{R}'^\times,\ \chi(a + (t^m)) = b + (t^m) \text{ in } \mathcal{R}'/(t^m)\}.$$

Ideally, we would like to extend the definition of $\omega_{m,r}$ to a map from $\Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times$ to $\Omega^1_{\underline{\mathcal{R}}/k}$. This can be done when $r = m + 1$ but it is not true if $m + 1 < r$.

However, it turns out that for us purposes, we do not need these 1-forms themselves but only their residues and we can construct these residues independently of all the choices. Suppose that $\mathcal{S}$ is a smooth k_m-algebra of relative dimension 1, with x a closed point and η the generic point of its spectrum. Suppose that $\mathcal{R}$, $\mathcal{R}'$ are liftings of $\mathcal{S}_\eta$ to k_r, with χ the corresponding isomorphism from $\mathcal{R}/(t^m)$ to $\mathcal{R}'/(t^m)$. We construct a map

$$res_x \omega_{m,r} : \Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times \to k',$$

where k' is the residue field of x, which is functorial and independent of all the choices. Let $\tilde{\chi} : \mathcal{R} \to \mathcal{R}'$ be an isomorphism of k_r-algebras which is a lifting of χ. Choosing also an isomorphism $\underline{\mathcal{R}}_r \simeq \mathcal{R}$ of k_r-algebras, provides us with an identification

$$(\mathcal{R}, \mathcal{R}', \chi)^\times \xrightarrow{\tilde{\chi}^*} (\underline{\mathcal{R}}_r, (t^m))^\times .$$

Let $\underline{\psi}$ denote the isomorphism $\underline{\mathcal{R}} \to \underline{\mathcal{S}}_\eta$ induced by the one from $\mathcal{R}/(t^m)$ to $\mathcal{S}_\eta$. Then we define $res_x \omega_{m,r}$ by the composition

$$\Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times \xrightarrow{\Lambda^3 \tilde{\chi}^*} \Lambda^3(\underline{\mathcal{R}}_r, (t^m))^\times \xrightarrow{\omega_{m,r}} \Omega^1_{\underline{\mathcal{R}}/k} \xrightarrow{d\underline{\psi}} \Omega^1_{\underline{\mathcal{S}}_\eta/k} \xrightarrow{res_x} k'.$$

We prove that this composition is independent of all the choices. Applying this construction in the above context, we see that $\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i})$ and $\tilde{\gamma}_{j_c}$ are two liftings of the same object γ_{j_c} to two different generic liftings of $\hat{\mathcal{O}}_{C,c}$. Therefore, the expression

$$res_c \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i,\eta}), \tilde{\gamma}_{j_c}) \in k(c)$$

is defined.

Applying traces and taking the sum over all the closed points, we obtain the expression in (1.3.1). Next we show that the sum is in fact a finite sum. The above construction involves many choices and would be completely useless if it depended on anything other than the initial data. This is the content of Theorem 8.1.1, our main theorem. Because of its basic properties that we prove below, $\rho_{m,r}$ deserves to be called a regulator.

Finally, let us mention that in [21], where the case $m = 2$ was handled, the only possible r is 3 and hence satisfies $r = m + 1$. In this case, the map $\omega_{2,3}$ can be defined as a map from $\Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times \to \Omega^1_{\underline{\mathcal{R}}/k}$. This is not true in general and this is why we have to pursue a different approach in this paper which is based on defining only the residue of the differential rather than the differential itself.

1.4. **Outline.** We give an outline of the paper. In §2, we describe the complex analytic version of our construction for motivation. In §3, we give a review of the construction in [18] of the additive dilogarithm on the Bloch group of a truncated polynomial ring. In §4, we describe the infinitesimal part of the Milnor K-theory of a local $\mathbb{Q}$-algebra endowed with a nilpotent ideal, which is split, in terms of Kähler differentials. Without any doubt the results in this section are known to the experts and we do not claim originality. The reason for our inclusion of this section is first that we could not find an easily quotable statement in the full generality which we will need in our later work, and second that we found a short argument which is in line with the general set-up of this paper. In §5, for a regular local $\mathbb{Q}$-algebra R, we define regulators $B_2(R_m) \otimes R_m^\times$ to Ω^1_R for every $m < r < 2m$, which vanishes on boundaries. This construction depends on the splitting of R_m in an essential way. In §6, we introduce the main object of this paper: for a smooth algebra R of relative dimension 1 over k, we define regulators $\omega_{m,r} : \Lambda^3(R_r, (t^m))^\times \to \Omega^1_{R/k}$, for each $m < r < 2m$. In §7, we compute the residues of the value of $\omega_{m,r}$ on good liftings. In §8, we use the results of the previous sections to construct the regulator from $\mathrm{H}^3_B(C, \mathbb{Q}(3))$ and specializing to triples of rational functions we obtain the infinitesimal Chow dilogarithm of higher modulus. In §9, we give examples of the infinitesimal Chow dilogarithm in the cases of the projective curve and elliptic curves and also give the applications to the strong reciprocity conjecture and the invariants of cycles.

Conventions and notation. We are interested in everything modulo torsion. Therefore, *we tensor all abelian groups under consideration with $\mathbb{Q}$ without explicitly signifying this in the notation.* For example, $K_n^M(A)$ denotes Milnor K-theory of A tensored with $\mathbb{Q}$ etc.

For a ring R, we let $R_\infty := R[[t]]$ (resp. $R((t))$) be the formal power series (resp. the formal Laurent series) ring over R. For $m \geq 1$, we let $R_m := R[t]/(t^m)$, be the truncated polynomial ring over R of modulus m. If R is a $\mathbb{Q}$-algebra then we write $\exp(\alpha) := \sum_{0 \leq n} \frac{\alpha^n}{n!}$ for $\alpha \in (t) \subseteq R_\infty$. The same formula is used for $\alpha \in (t) \subseteq R_m$.

For an appropriate functor F, we let $F(R_\infty)^\circ := ker(F(R_\infty) \to F(R))$ (resp. $F(R_m)^\circ := ker(F(R_m) \to F(R))$), denote the infinitesimal part of $F(R_\infty)$ (resp. $F(R_m)$).

For any set X, let $\mathbb{Q}[X]$ denote the vector space over $\mathbb{Q}$ with basis $\{[x]|x \in X\}$.

We denote both the differential $B_2(A) \to \Lambda^2 A^\times$ in the Bloch complex of weight 2 (cf. §2.1.1) and the differential $B_2(A) \otimes A^\times \to \Lambda^3 A^\times$ in the Bloch complex of weight 3 (cf. §2.1.2) by δ. Since the sources of the maps are different, this will not cause any confusion. When we use these maps in the case when $A = R_r$ and we want to emphasize dependence on r in the notation, we denote both of these differentials by δ_r.

2. The analogy with the complex case

In this section we will describe the analogy with the complex case after recalling some of the standard definitions. Our aim is to give a flavor of the concepts before going into the technical details.

2.1. Basic definitions.

Here we collect some of the basic definitions that are standard in the literature. We will use these definitions in this section and generalize them in the later sections.

2.1.1. *The Bloch group B_2 and the Bloch complex of weight 2.* For any ring A, we let $A^\flat := \{a \in A|a(1-a) \in A^\times\}$. For a local $\mathbb{Q}$-algebra R, the Bloch group $B_2(R)$ is the quotient of $\mathbb{Q}[R^\flat]$ by the subspace generated by

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)],$$

for all $x, y \in R^\flat$ such that $x - y \in R^\times$. There is a map $\delta : B_2(R) \to \Lambda^2 R^\times$, which is defined on the generators by letting $\delta([x]) := (1-x) \wedge x$. The corresponding complex obtained by putting $B_2(R)$ in degree 1 and $\Lambda^2 R^\times$ is degree 2 is called *the Bloch complex of weight 2.* This complex computes the weight 2 motivic cohomology of R, when R is a field. We refer to [18] for details about the Bloch group and the Bloch complex of weight 2. We denote the cohomology of this complex with $\mathrm{H}^i(R, \mathbb{Q}(2))$.

2.1.2. *The pre-Bloch complex of weight 3.* Continuing with the notation above, we have a complex

$$\mathbb{Q}[R^\flat] \to B_2(R) \otimes R^\times \to \Lambda^3 R^\times, \qquad (2.1.1)$$

concentrated in degrees $[1, 3]$, where the first map sends a basis element $[x]$ to $[x] \otimes x$ and the second one sends $[x] \otimes y$ to $\delta(x) \wedge y$. Abusing the notation, we denote all the differentials in this complex by δ. This complex and its variants are defined and studied in detail in [6]. We will call this complex *the pre-Bloch complex of weight 3.*

The first group $\mathbb{Q}[R^\flat]$ when divided by the appropriate relations is denoted by $B_3(R)$. At this stage of this theory, the exact type of these relations are not clear. There are several different candidates and it is not known that they give the same answer [6]. The corresponding sequence obtained is a candidate for the weight 3 motivic cohomology complex [6]. Since we will only deal with the cohomology groups in degrees 2 and 3, we will only work with the pre-Bloch complex (2.1.1) above and the precise relations in order to define $B_3(R)$ will not be important for us. The reason that we do not call this complex the Bloch complex is that we do not use a version of the group $B_3(R)$ and instead use $\mathbb{Q}[R^\flat]$. For R equal to the dual numbers of a field, this complex and its higher weight analogs were used in [19] to construct the additive polylogarithms.

For the degrees $i = 2$ and 3, we will denote the cohomology of the pre-Bloch complex in (2.1.1) by $\mathrm{H}^i(R, \mathbb{Q}(3))$.

2.1.3. *Residue map between the Bloch complexes.* Suppose that R is a discrete valuation ring with residue field k and with field of fractions K. There is a canonical residue homomorphism $K_n^M(K) \to K_{n-1}^M(k)$ constructed by Milnor [13] between the Milnor K-groups. Goncharov generalized to a map between the Bloch complexes in [6, §1.14].

For us, the only parts of this construction that will be relevant are:

$$res : \Lambda^n K^\times \to \Lambda^{n-1} k^\times$$

and

$$res : B_2(K) \otimes K^\times \to B_2(k).$$

To describe these maps, let us fix a uniformizer π of R. The map will turn out to be independent of the choice of the uniformizer.

The first map is determined by the following formula

$$res(u_0 \pi^m \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_{n-1}) = m \cdot \underline{u}_1 \wedge \underline{u}_2 \wedge \cdots \wedge \underline{u}_{n-1},$$

where $m \in \mathbb{Z}$, u_i, for $1 \leq i < n$ are units in R and $\underline{u}_i$ for $1 \leq i < n$ are the images of u_i in k.

The second map is determined by the formulas that

$$res([a] \otimes b) = 0,$$

if $a \in K^\flat \setminus R^\flat$ or $b \in R^\times$; and

$$res([u] \otimes \pi) = [\underline{u}],$$

if $u \in R^\flat$ and $\underline{u}$ is the image of u in $k^\flat$.

These maps give a commutative diagram

$$\begin{array}{ccc} B_2(K) \otimes K^\times & \longrightarrow & \Lambda^3 K^\times \\ \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle res} \\ B_2(k) & \longrightarrow & \Lambda^2 k^\times \end{array}$$

and hence a sequence

$$B_2(K) \otimes K^\times \to B_2(k) \oplus \Lambda^3 K^\times \to \Lambda^2 k^\times.$$

If we start with a smooth curve X/k, then taking residues at all the closed points and summing them will give a sequence

$$B_2(k(X)) \otimes k(X)^\times \to \oplus_{x \in |X|} B_2(k(x)) \oplus \Lambda^3 k(X)^\times \to \oplus_{x \in |X|} \Lambda^2 k(x)^\times.$$

This is part of the motivic complex of weight 3 of the curve X [6], whose middle cohomology receives a map from the motivic cohomology $\mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3))$ of X.

2.2. **Complex analog of the main construction.** Here we briefly explain the complex analog of our construction, which is one of our main motivations for the infinitesimal case. If $X/\mathbb{C}$ is a smooth projective curve, then as above one expects a map

$$K_3(X)^{(3)}_{\mathbb{Q}} = \mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3)) \to \mathrm{H}^1_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(2)) = K_3(\mathbb{C})^{(2)}_{\mathbb{Q}}.$$

Composing with the Borel regulator $K_3(\mathbb{C})^{(2)}_{\mathbb{Q}} \to \mathbb{C}/(2\pi i)^2\mathbb{Q}$ and taking the imaginary part would give a map $K_3(X)^{(3)}_{\mathbb{Q}} \to \mathbb{R}$. Up to normalization, this map can be constructed as follows [7, §6]. For f_1, f_2, and $f_3 \in \mathbb{C}(X)^\times$, let

$$r_2(f_1, f_2, f_3) := \mathrm{Alt}_3\Big(\frac{1}{6} \log|f_1| \cdot d\log|f_2| \wedge d\log|f_3| - \frac{1}{2} \log|f_1| \cdot d\arg f_2 \wedge d\arg f_3\Big),$$

(such that $dr_2(f_1, f_2, f_3) = \mathrm{Re}(d\log(f_1) \wedge d\log(f_2) \wedge d\log(f_3))$). The Chow dilogarithm map $\rho : \Lambda^3 \mathbb{C}(X)^\times \to \mathbb{R}$ is given in terms of this by

$$\rho(f_1 \wedge f_2 \wedge f_3) := \int_{X(\mathbb{C})} r_2(f_1, f_2, f_3). \tag{2.2.1}$$

In the special case when $X = \mathbb{P}^1$, we have

$$\rho((1-z) \wedge z \wedge (z-a)) = D_2(a), \tag{2.2.2}$$

where $D_2(z) := \mathrm{Im}(\ell i_2(z)) + \arg(1-z) \cdot \log(|z|)$ is the Bloch-Wigner dilogarithm, with $\ell i_2(z)$ the (multi-valued) analytic continuation of $\sum_{1\leq n} \frac{z^n}{n^2}$.

The middle cohomology of

$$(2.2.3) \qquad \to B_2(\mathbb{C}(X)) \otimes \mathbb{C}(X)^{\times}_{\mathbb{Q}} \to (\oplus_{x\in X} B_2(\mathbb{C})) \oplus \Lambda^3 \mathbb{C}(X)^{\times}_{\mathbb{Q}} \to \oplus_{x\in X} \Lambda^2 \mathbb{C}^{\times}_{\mathbb{Q}} \to$$

receives a map from $\mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3)) \simeq K_3(X)^{(3)}_{\mathbb{Q}}$. Combining D_2 and ρ, if we let

$$\rho_X := -(\oplus_{x\in X} D_2) \oplus \rho : (\oplus_{x\in X} B_2(\mathbb{C})) \oplus \Lambda^3 \mathbb{C}(X)^{\times}_{\mathbb{Q}} \to \mathbb{R},$$

then ρ_X vanishes on the image of $B_2(\mathbb{C}(X)) \otimes \mathbb{C}(X)^{\times}_{\mathbb{Q}}$ and induces the map $K_3(X)^{(3)}_{\mathbb{Q}} \to \mathbb{R}$, we were looking for above. If one assumes a theory of motivic sheaves then this is the composition of $\mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3)) \to \mathrm{Ext}^1_{\mathcal{M}_{\mathbb{C}}}(\mathbb{Q}(0), \mathrm{H}^2(X/\mathbb{C})(3)) = \mathrm{H}^1_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(2)) \to B_2(\mathbb{C})$ and the Bloch-Wigner dilogarithm $D_2 : B_2(\mathbb{C}) \to \mathbb{R}$. Here $\mathcal{M}_{\mathbb{C}}$ denotes the category of motives over $\mathbb{C}$ and $\mathrm{H}^2(X/\mathbb{C}) = \mathbb{Q}(-1)$ denotes the relative motivic cohomology of $X/\mathbb{C}$.

In the special case of $X = \mathbb{P}^1$ the map ρ can be made even more explicit [7, §6.3]. Suppose that f_1, f_2, and f_3 are arbitrary rational functions on $\mathbb{P}^1$. By the linearity of ρ and the fact that ρ vanishes on elements of the from $\lambda \wedge f \wedge g$, if $\lambda \in \mathbb{C}^{\times}$, we notice that in order to determine $\rho(f_1 \wedge f_2 \wedge f_3)$, it is enough to determine its value for $f_i = z - \alpha_i$, for pairwise distinct α_i. Using functoriality with respect to automorphisms of $\mathbb{P}^1$ fixing ∞, and the formula (2.2.2), we determine that

$$(2.2.4) \qquad \rho((z-\alpha_1) \wedge (z-\alpha_2) \wedge (z-\alpha_3)) = D_2(\frac{\alpha_3 - \alpha_2}{\alpha_1 - \alpha_2}).$$

3. Additive dilogarithm of higher modulus

In this section, we review and rephrase the theory of the additive dilogarithm over truncated polynomial rings in a manner which we will need in the remainder of the paper. Further results for this function can be found in [18].

For a $\mathbb{Q}$-algebra R, let $R_\infty := R[[t]]$, denote the formal power series over R and $R_m := R_\infty/(t^m)$ the truncated polynomial ring of modulus m over R. Since R is a $\mathbb{Q}$-algebra we have the logarithm $\log : (1 + tR_\infty)^\times \to R_\infty$ given by $\log(1+z) := \sum_{1\leq n} (-1)^{n+1} \frac{z^n}{n}$, for $z \in tR_\infty$. Let $\log^\circ : R_\infty^\times \to R_\infty$, be the branch of the logarithm associated to the splitting of $R_\infty \twoheadrightarrow R$ corresponding to the inclusion $R \hookrightarrow R_\infty$, defined as $\log^\circ(\alpha) := \log(\frac{\alpha}{\alpha(0)})$. If $q = \sum_{0\leq i} q_i t^i \in R_\infty$ and $1 \leq a$ then let $q|_a := \sum_{0\leq i<a} q_i t^i \in R_\infty$, denote the truncation of q to the sum of the first a-terms, and $t_a(q) := q_a$, the coefficient of t^a in q. If $u \in tR_\infty$ and $s(1-s) \in R^\times$, we let

$$(3.0.1) \qquad \ell i_{m,r}(s\exp(u)) := t_{r-1}(\log^\circ(1 - s\exp(u|_m)) \cdot \frac{\partial u}{\partial t}\Big|_{r-m}),$$

for $m < r < 2m$. Here, and everywhere in the paper, $\exp(z)$ denotes the formal power series $\sum_{0\leq n} \frac{z^n}{n!}$. Also note that $\frac{\partial u}{\partial t}\big|_{r-m}$ denotes the truncation of the derivative of u with respect to t, to the sum of its first $(r-m)$-terms. Fixing $m \geq 2$, these $\ell i_{m,r}$'s, for $m < r < 2m$ together constitute a regulator for the kernel of the map from $K_3(R_m)^{(2)}$ to $K_3(R)^{(2)}$. This is exactly analogous to the Bloch-Wigner dilogarithm in the complex case [1], [17], [18].

Since every element of $R_\infty^\flat$ can be written in the form $s\exp(u)$ as the above, we can linearly extend $\ell i_{m,r}$, to obtain a map from the vector space $\mathbb{Q}[R_\infty^\flat]$ with basis $R_\infty^\flat$. We denote this map by the same symbol.

When we would like to specify the δ defined on $B_2(R_\infty)$ (resp. $B_2(R_m)$), as given in §2.1.1, we denote it by δ_∞ (resp. δ_m).

Let V be a free R module with basis $\{e_i\}_{i\in I}$ and $\{e_i^\vee\}_{i\in I}$ the dual basis of $V^\vee$. Given v and $\alpha = \sum_{i\in I} a_i e_i$ in V, we let

$$(v|\alpha) := \sum_{i\in I} a_i e_i^\vee(v) \in R.$$

If there is an ordering on I, we let $\{e_i \wedge e_j\}_{i>j}$ be the corresponding basis of $\Lambda^2 V$. Then, with the above notation, the expression $(w|\beta)$, for w, $\beta \in \Lambda^2 V$, is defined. We consider tR_∞, as a free R-module with basis $\{t^i\}_{1\leq i}$.

Let us denote the composition of $B_2(R_\infty) \xrightarrow{\delta} \Lambda^2 R_\infty^\times$ with the canonical projection $\mathbb{Q}[R_\infty^\flat] \to B_2(R_\infty)$ also by δ. Also denote the map

$$\Lambda^2 R_\infty^\times \to \Lambda^2 tR_\infty \twoheadrightarrow \Lambda^2_R tR_\infty$$

induced by $\Lambda^2 \log^\circ : \Lambda^2 R_\infty^\times \to \Lambda^2 tR_\infty$, by the same symbol.

Proposition 3.0.1. *With the notation above, for $\alpha \in \mathbb{Q}[R_\infty^\flat]$ and $2 \leq m < r < 2m$, we have*

$$\ell i_{m,r}(\alpha) = \Big(\Lambda^2 \log^\circ(\delta(\alpha))| \sum_{1\leq i\leq r-m} it^{r-i} \wedge t^i\Big), \tag{3.0.2}$$

and this function descends through the canonical projections

$$\mathbb{Q}[R_\infty^\flat] \to B_2(R_\infty) \to B_2(R_m),$$

to define a map from $B_2(R_m)$ to R, denoted by the same notation.

Proof. We proved in [18, Prop. 2.2.1] that the function defined by the right hand side of (3.0.2), temporarily denote it by $\ell i^*_{m,r}$, descends to give a map from $\mathbb{Q}[R_m^\flat]$ and in [18, Prop. 2.2.2] that it descends to give a map from $B_2(R_m)$. Therefore it only remains to prove the equality (3.0.2).

With the notation $\ell_i(\alpha) := t_i(\log^\circ(\alpha))$, $\ell i^*_{m,r}$ can be rewritten as

$$\ell i^*_{m,r} = \Big(\sum_{1\leq i\leq r-m} i \cdot \ell_{r-i} \wedge \ell_i\Big) \circ \delta.$$

Then we have $\ell i^*_{m,r}(s\exp(u)) = \ell i^*_{m,r}(s\exp(u|_m))$, since we know that $\ell i^*_{m,r}$ descends to $\mathbb{Q}[R_m^\flat]$. We have $\ell_i(s\exp(u|_m)) = u_i$, for $1 \leq i < m$ and $\ell_i(s\exp(u|_m)) = 0$, for $m \leq i$. Using this we obtain that $\ell i^*_{m,r}(s\exp(u|_m)) = \sum_{1\leq i\leq r-m} i \cdot \ell_{r-i}(1 - s\exp(u|_m)) \cdot u_i = \ell i_{m,r}(s\exp(u))$. □

Let us give a name to the essential map which constitute $\ell i_{m,r}$.

Definition 3.0.2. *We denote the map from $\Lambda^2 R_\infty^\times$ to R which sends $\alpha \wedge \beta$ to*

$$(\Lambda^2 \log^\circ(\alpha \wedge \beta)| \sum_{1\leq i\leq r-m} it^{r-i} \wedge t^i)$$

by $\ell_{m,r}$. It is clear that $\ell_{m,r} : \Lambda^2 R_\infty^\times \to R$ factors through the projection $\Lambda^2 R_\infty^\times \to \Lambda^2 R_r^\times$. The additive dilogarithm above is given in terms of this function as

$$\ell i_{m,r} = \ell_{m,r} \circ \delta_\infty = \ell_{m,r} \circ \delta_r.$$

We will use the main result from [18], there it was stated in the case when R is a field of characteristic 0, but the same proof works when R is a regular, local $\mathbb{Q}$-algebra. Let $B_2(R_m)^\circ$ denote the kernel of the natural map from $B_2(R_m)$ to $B_2(R)$, consistent with the notation in the introduction.

Theorem 3.0.3. *The complex $B_2(R_m)^\circ \xrightarrow{\delta^\circ} (\Lambda^2 R_m^\times)^\circ$ computes the infinitesimal part of the weight two motivic cohomology of R_m, and the map $\oplus_{m<r<2m} \ell i_{m,r}$ induces an isomorphism*

$$\mathrm{HC}_2^\circ(R_m)^{(1)} \simeq K_3^\circ(R_m)^{(2)} \simeq ker(\delta^\circ) \xrightarrow{\sim} R^{\oplus(m-1)}$$

from the relative cyclic homology group $\mathrm{HC}_2^\circ(R_m)^{(1)}$ to $R^{\oplus(m-1)}$.

4. Infinitesimal Milnor K-theory of local rings

Suppose that R is a local $\mathbb{Q}$-algebra and A is an R-algebra, together with a nilpotent ideal I such that the natural map $R \to A/I$ is an isomorphism. Then the Milnor K-theory $K_n^M(A)$ of A, naturally splits into a direct sum $K_n^M(A) = K_n^M(R) \oplus K_n^M(A)^\circ$. In this section, we will describe this infinitesimal part $K_n^M(A)^\circ$ in terms of Kähler differentials. It is easy to find such an isomorphism using Goodwillie's theorem [8], and standard computations in cyclic homology. However, in the next section, we need an explicit description of this isomorphism in order to determine which symbols vanish in the corresponding Milnor K-group. Fortunately, determining what this isomorphism turns out to be quite easy. By the functoriality and the multiplicativity of the isomorphism, we reduce the computation to the case of K_2^M of the dual numbers over R where the computation is easy.

There is no doubt that the results in this section are well-known and we do not claim any originality. We simply have not been able to find a description of the map φ below which is easily quotable in the literature. Since our discussion is quite short we did not refrain from including it in the present paper. We will only need the result below for $A = R_m$. On the other hand, in a future work we will need this result in full generality which justifies our somewhat more general discussion:

Proposition 4.0.1. *There exists a unique map $\varphi : K_n^M(A)^\circ \to \Omega_A^{n-1}/(d\Omega_A^{n-2} + \Omega_R^{n-1})$ such that*

$$\varphi(\{\alpha, \beta_1, \cdots, \beta_{n-1}\}) = \log(\alpha)\frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}, \tag{4.0.1}$$

for $\alpha \in 1 + I$ and $\beta_1, \cdots, \beta_{n-1} \in A^\times$, and this map is an isomorphism.

Proof. The uniqueness follows from the fact that the infinitesimal part of Milnor K-theory is generated by terms $\{\alpha, \beta_1, \cdots, \beta_{n-1}\}$ as in the statement. In order to see this, let $\iota : R \to A$ denote the structure map. Since $R \to A \to A/I$ is an isomorphism, every element in $A^\times$ can be uniquely written as $\iota(r)\alpha$ with $r \in R^\times$ and $\alpha \in 1 + I$. This implies that $K_n^M(A)$ is generated by elements of the form $\{\alpha_1, \cdots, \alpha_i, \iota(r_1), \cdots, \iota(r_{n-i})\}$, with $0 \leq i \leq n$ and $r_j \in R$, $\alpha_j \in 1 + I$. The terms with $1 \leq i$ are in $K_n^M(A)^\circ = \ker(K_n^M(A) \to K_n^M(A/I))$. They are also of the form $\{\alpha, \beta_1, \cdots, \beta_{n-1}\}$. In order to prove the statement, we only need to show that if a linear combination of terms of the form $\{\iota(r_1), \cdots, \iota(r_n)\}$ are in $\ker(K_n^M(A) \to K_n^M(A/I))$ then it is in fact 0. This again follows from the fact that the natural map from R to A/I is an isomorphism.

We define a functorial map φ by the following composition:

$$K_n^M(A)^\circ \to K_n^{(n)}(A)^\circ \xrightarrow{\sim} \mathrm{HC}_{n-1}^{(n-1)}(A)^\circ = (\Omega_A^{n-1}/d\Omega_A^{n-2})^\circ = \Omega_A^{n-1}/(d\Omega_A^{n-2} + \Omega_R^{n-1}). \tag{4.0.2}$$

The first map is the multiplicative map induced by the isomorphism when $n = 1$, the second one is the Goodwillie isomorphism [8], and the last one is given by [12, Theorem 4.6.8].

By Nesterenko-Suslin's theorem [14], Milnor K-theory is the first obstruction to the stability of the homology of general linear groups:

$$K_n^M(A) \simeq \mathrm{H}_n(\mathrm{GL}_n(A), \mathbb{Q})/\mathrm{H}_n(\mathrm{GL}_{n-1}(A), \mathbb{Q}).$$

Moreover, the composition

$$K_n^M(A)^\circ \to K_n^{(n)}(A)^\circ \to Prim(\mathrm{H}_n(\mathrm{GL}(A), \mathbb{Q})) \to \mathrm{H}_n(\mathrm{GL}(A), \mathbb{Q}) \simeq \mathrm{H}_n(\mathrm{GL}_n(A), \mathbb{Q}) \twoheadrightarrow K_n^M(A)$$

is multiplication by $(n-1)!$ by [14]. This implies the injectivity of φ. It only remains to prove the property (4.0.1), since then the surjectivity of φ also follows.

The multiplicativity of φ takes the following form: for a, $b \in K_m^M(A)^\circ$, $\varphi(a \cdot b) = \varphi(a) \wedge d(\varphi(b))$. We do induction on n. The statement is clear for $n = 1$. We show that we may assume that $\beta_i \in R^\times$:

Lemma 4.0.2. *Suppose that we have the formula (4.0.1) for $\alpha \in 1 + I$ and $\beta_i \in R^\times$, for $1 \leq i \leq n-1$, then we have the same formula for $\alpha \in 1 + I$ and $\beta_i \in A^\times$, for $1 \leq i \leq n-1$.*

Proof. We do induction on the number of β_i which are not in $R^\times$. If all of them are in $R^\times$, the hypothesis of the lemma gives the expression. If there is at least one β_i which is not in $R^\times$, without loss of generality assume that $\beta_{n-1} \notin R^\times$. Let us write $\beta_{n-1} := \lambda \cdot \beta$, with $\lambda \in R^\times$ and $\beta \in 1 + I$. Then

$$\varphi(\{\alpha, \beta_1, \cdots, \beta_{n-1}\}) = \varphi(\{\alpha, \beta_1, \cdots, \beta_{n-2}, \lambda\}) + \varphi(\{\alpha, \beta_1, \cdots, \beta_{n-2}, \beta\}).$$

By the multiplicativity of φ, the formula for $n = 1$, and the induction hypothesis on n, we have

$$\varphi(\{\alpha, \beta_1, \cdots, \beta_{n-2}, \beta\}) = \varphi(\{\alpha, \beta_1, \cdots, \beta_{n-2}\}) \wedge d(\log(\beta)) = \log(\alpha)\frac{d\beta_1}{\beta_1} \wedge \cdots \frac{d\beta_{n-2}}{\beta_{n-2}} \wedge \frac{d\beta}{\beta}.$$

By the induction hypothesis on the number of β_i not in $R^\times$, we have

$$\varphi(\{\alpha, \beta_1, \cdots, \beta_{m-1}, \lambda\}) = \log(\alpha)\frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{m-1}}{\beta_{m-1}} \wedge \frac{d\lambda}{\lambda}.$$

Adding these two expressions, we obtain the expression we were looking for. □

The above lemma shows that we may without loss of generality assume that the $\beta_i \in R^\times$. The next lemma shows that we may also assume that $A = R_r$ and $\alpha = 1 + t$.

Lemma 4.0.3. *Suppose that we have the formula (4.0.1) for $\alpha = 1 + t$ and $\beta_i \in R^\times$ for $1 \le i \le n-1$, for the ring $R_r := R[t]/(t^r)$. Then we have the same formula for any A as above.*

Proof. Given $\alpha \in 1 + I \subseteq A^\times$ and $\beta_i \in R^\times$. Since $\alpha - 1$ is nilpotent, we have an R-algebra morphism $\psi : R_r \to A$, for some r, such that $\psi(t) = \alpha - 1$. The result then follows by the functoriality of φ since the map induced by ψ maps $\{1+t, \beta_1, \cdots, \beta_{n-1}\}$ to $\{\alpha, \beta_1, \cdots, \beta_{n-1}\}$. □

Next we will show that we can also assume that $r = 2$. Note that for each $\lambda \in \mathbb{Q}^\times$, we obtain an R-automorphism ψ_λ of R_r which sends t to $\lambda \cdot t$. If F is any functor from the category rings to the category of $\mathbb{Q}$-vector spaces, this gives us an action of $\mathbb{Q}^\times$ on $F(R_r)$, which we call the $\star$-action of $\mathbb{Q}^\times$ and denote $F(\psi_\lambda)(v)$ by $\lambda \star v$. For $m \in \mathbb{Z}$, we let $F(R_r)^{[m]}$ denote the subspace of elements $v \in F(R_r)$ such that $\lambda \star v = \lambda^m \cdot v$, for every $\lambda \in \mathbb{Q}^\times$. An element $v \in F(R_r)^{[m]}$ is said to be an element of $\star$-weight m.

Lemma 4.0.4. *Suppose that we have the formula (4.0.1) for $\alpha = 1 + t$ and $\beta_i \in R^\times$ for $1 \le i \le n-1$, for the ring R_2. Then we have the same formula for any A as above.*

Proof. We need to prove the result for $1 + t \in R_r$, and $\beta_i \in R^\times$. Since $1 + t = \exp(\log(1 + t))$, it is a product of elements of the form $\exp(at^m)$, for $1 \le m < r$, and $a \in \mathbb{Q}$. Therefore it is enough to prove the formula for elements as above with $\alpha = \exp(at^m)$.

Since $\{\exp(at^m), \beta_1, \cdots, \beta_{n-1}\}$ is of $\star$-weight m, its image under φ is in $(\Omega^{n-1}_{R_r}/d\Omega^{n-2}_{R_r})^{[m]}$, the $\star$-weight m part of $\Omega^{n-1}_{R_r}/d\Omega^{n-2}_{R_r}$. On the other hand the natural surjection $R_r \to R_{m+1}$ induces an isomorphism

$$(\Omega^{n-1}_{R_r}/d\Omega^{n-2}_{R_r})^{[m]} \simeq (\Omega^{n-1}_{R_{m+1}}/d\Omega^{n-2}_{R_{m+1}})^{[m]}.$$

Therefore, without loss of generality, we will assume that $r = m + 1$. Then we use the map from R_2 to R_{m+1} that sends t to at^m. This map sends $1 + t$ to $\exp(at^m)$ and hence maps $\{1 + t, \beta_1, \cdots, \beta_{n-1}\}$ to $\{\exp(at^m), \beta_1, \cdots, \beta_{n-1}\}$. Therefore, again by the functoriality of φ, the result follows from the assumption on R_2. □

To finish the proof, we will need a special identity in $K_2^M(R_3)$:

Lemma 4.0.5. *We have the following relation in $K_2^M(R_3)$:*

$$2\{1 + \frac{t^2}{2}, \lambda\} = \{1 + \frac{t}{\lambda}, 1 + \lambda t\},$$

for any $\lambda \in R^\times$.

Proof. It is possible to give a direct computational proof of this statement. We choose to give a proof which is based on the ideas in this section.

First suppose that R is a field. We know that both sides are in $K_2^M(R_3)^\circ$. We know from [9] that the map $K_2^M(R_3)^\circ \to (\Omega^1_{R_3}/dR_3)^\circ$ which sends $\{\alpha, \beta\}$ to $\log(\alpha)\frac{d\beta}{\beta}$, where $\alpha - 1 \in (t)$, is an isomorphism.

The left hand side goes to $t^2\frac{d\lambda}{\lambda}$, whereas the right hand side goes to

$$\frac{t}{\lambda}d(\lambda t) = t^2\frac{d\lambda}{\lambda} + tdt = t^2\frac{d\lambda}{\lambda} + \frac{1}{2}dt^2 = t^2\frac{d\lambda}{\lambda}$$

in $(\Omega^1_{R_3}/dR_3)^\circ$. This proves the statement when R is a field.

In general, the statement for $\mathbb{Q}[x, x^{-1}]$ implies the one for a general R by sending x to λ. Finally, if we can show that $K_2^M(\mathbb{Q}[x, x^{-1}]_3)^\circ \to K_2^M(\mathbb{Q}(x)_3)^\circ$ is an injection, the known statement for

$\mathbb{Q}(x)$ implies the one for $\mathbb{Q}[x, x^{-1}]$. This injectivity follows from the commutative diagram

$$\begin{array}{ccccc} K_2^M(\mathbb{Q}[x,x^{-1}]_3)^\circ & \xhookrightarrow{\varphi} & (\Omega^1_{\mathbb{Q}[x,x^{-1}]_3}/d(\mathbb{Q}[x,x^{-1}]_3))^\circ & \xleftarrow[\sim]{} & t\Omega^1_{\mathbb{Q}[x,x^{-1}]} \oplus t^2\Omega^1_{\mathbb{Q}[x,x^{-1}]} \\ \downarrow & & \downarrow & & \downarrow \\ K_2^M(\mathbb{Q}(x)_3)^\circ & \xhookrightarrow{\varphi} & (\Omega^1_{\mathbb{Q}(x)_3}/d(\mathbb{Q}(x)_3))^\circ & \xleftarrow[\sim]{} & t\Omega^1_{\mathbb{Q}(x)} \oplus t^2\Omega^1_{\mathbb{Q}(x)}, \end{array}$$

where the injectivity of φ was proven above. This finishes the proof of the lemma. □

Finally, we prove the result for R_2.

Proposition 4.0.6. *Let $\alpha = 1+t \in R_2^\times$ and $\beta_i \in R^\times$, for $1 \le i \le n-1$, then $\varphi(\{\alpha, \beta_1, \cdots, \beta_{n-1})$ is given by (4.0.1).*

Proof. Note the map ψ from R_2 to R_3 that sends t to $\frac{t^2}{2}$. This map induces an isomorphism

$$(\Omega^{n-1}_{R_2}/d\Omega^{n-2}_{R_2})^{[1]} \simeq (\Omega^{n-1}_{R_3}/d\Omega^{n-2}_{R_3})^{[2]}.$$

Therefore we only need to compute the image of

$$\{1 + \frac{t^2}{2}, \beta_1, \cdots, \beta_{n-1}\} \tag{4.0.3}$$

in $\Omega^{n-1}_{R_3}/d\Omega^{n-2}_{R_3})^{[2]}$. By the previous lemma, we know that

$$\{1 + \frac{t^2}{2}, \beta_1\} = \frac{1}{2}\{1 + \frac{t}{\beta_1}, 1 + \beta_1 t\},$$

which implies that (4.0.3) is equal to

$$\frac{1}{2}\{1 + \frac{t}{\beta_1}, 1 + \beta_1 t, \beta_2, \cdots, \beta_{n-1}\}. \tag{4.0.4}$$

This last expression is the $\frac{1}{2}$ times the product of $\{1 + \frac{t}{\beta_1}\} \in K_1^M(R_3)^\circ$ and

$$\{1 + \beta_1 t, \beta_2, \cdots, \beta_{n-1}\} \in K^M_{n-1}(R_3)^\circ.$$

By the induction hypothesis on n,

$$\varphi(\{1 + \beta_1 t, \beta_2, \cdots, \beta_{n-1}\}) = \log(1 + \beta_1 t)\frac{d\beta_2}{\beta_2} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}.$$

Since φ is multiplicative, this implies that (4.0.4) is sent by φ to

$$\frac{1}{2}\log(1 + \frac{t}{\beta_1})d\log(1 + \beta_1 t)\frac{d\beta_2}{\beta_2} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}} = \log(1 + \frac{t^2}{2})\frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}.$$

□

This finishes the proof of Proposition 4.0.1. □

In the case of truncated polynomial rings, we can also describe this isomorphism as follows:

Corollary 4.0.7. *The map $\lambda_i : \Lambda^n R^\times_\infty \to \Omega^{n-1}_R$ given by*

$$\lambda_i(a_1 \wedge \cdots \wedge a_n) = res_{t=0}\frac{1}{t^i}d\log(a_1) \wedge \cdots \wedge d\log(a_n) \in \Omega^{n-1}_R,$$

for $1 \le i < r$, descends to give a map $K^M_n(R_r)^\circ \to \Omega^{n-1}_R$. Their sums induce an isomorphism:

$$K^M_n(R_r)^\circ \to \oplus_{1 \le i < r}\Omega^{n-1}_R.$$

Proof. For $1 \leq i < r$, we let $\mu_i : (\Omega^{n-1}_{R_r}/d(\Omega^{n-2}_{R_r}))^\circ \to \Omega^{n-1}_R$ be given by $\mu_i(w) := res_{t=0}\frac{1}{t^i}d\omega$. The induced map

$$(\Omega^{n-1}_{R_r}/d(\Omega^{n-2}_{R_r}))^\circ \to \oplus_{1\leq i<r}\Omega^{n-1}_R$$

is an isomorphism.

The surjectivity can be seen as follows. Given $\omega \in \Omega^{n-1}_R$, and $1 \leq i,\, j < r$,

$$\mu_j(t^i\omega) = res_{t=0}\frac{1}{t^j}d(t^i\omega) = \delta_{ij}\cdot i\cdot\omega.$$

In order to prove injectivity, using the notation in the proof of Lemma 4.0.3, we note that $(\Omega^{n-1}_{R_r}/d(\Omega^{n-2}_{R_r}))^\circ$ is the direct sum of its subspaces $((\Omega^{n-1}_{R_r}/d(\Omega^{n-2}_{R_r}))^\circ)^{[i]}$ of $\star$-weight i, for $1 \leq i < r$. The subspace $((\Omega^{n-1}_{R_r}/d(\Omega^{n-2}_{R_r}))^\circ)^{[i]}$ consists of elements of the form $t^i\alpha + \beta t^{i-1}dt$, with $\alpha \in \Omega^{n-1}_R$ and $\beta \in \Omega^{n-2}_R$. For $1 \leq j < r$, we have

$$\mu_j(t^i\alpha + \beta t^{i-1}dt) = \delta_{ij}(i\cdot\alpha + (-1)^{n-1}d\beta).$$

Therefore, if $\mu_i(t^i\alpha + \beta t^{i-1}dt) = 0$, then $\alpha = \frac{(-1)^n}{i}d\beta$ and hence

$$t^i\alpha + \beta t^{i-1}dt = d(\frac{(-1)^n\beta}{i}t^i).$$

This proves the injectivity of the map

$$(\Omega^{n-1}_{R_r}/d(\Omega^{n-2}_{R_r}))^\circ = \oplus_{1\leq i<r}((\Omega^{n-1}_{R_r}/d(\Omega^{n-2}_{R_r}))^\circ)^{[i]} \to \oplus_{1\leq i<r}\Omega^{n-1}_R.$$

The corollary then follows from Proposition 4.0.1. □

5. Construction of maps from $B_2(R_m)\otimes R_m^\times$ to Ω^1_R

In this section, we assume that R is a regular, local $\mathbb{Q}$-algebra.

5.1. **Preliminaries on the construction.** In this section, we fix m and r such that $2 \leq m < r < 2m$. We let $f(s,u) := \log^\circ(1 - s\exp(u)) = \log(\frac{1-s\exp(u)}{1-s})$. As in the proof of Proposition 3.0.1, we define $\ell_i : R^\times_\infty \to R$, by the formula $\ell_i(a) := t_i(\log^\circ(a))$. Note that t_i is defined in the beginning of §3. Let us consider the expression

$$\alpha_j := \sum_{1\leq i\leq j-1} id\ell_{j-i}\wedge\ell_i = \sum_{\substack{a+b=j\\ 1\leq a,\, b}} bd\ell_a\wedge\ell_b, \tag{5.1.1}$$

for $m \leq j < r$, which defines a map from $\Lambda^2 R^\times_\infty$ to Ω^1_R. We will use this expression to define a map from $B_2(R_m)\otimes R_m^\times$.

Lemma 5.1.1. *For $s \in R^\flat$, and $u := \sum_{0<i}u_it^i \in tR_\infty$, letting $f_s := \frac{\partial f}{\partial s}$ and $u_t := \frac{\partial u}{\partial t} = \sum_{0<i}iu_it^{i-1}$, we have*

$$\alpha_j(\delta(s\exp(u))) = t_{j-1}(f_su_t)ds = t_{j-1}(\frac{\partial\log^\circ(1-s\exp(u))}{\partial s}\cdot\frac{\partial u}{\partial t})ds.$$

Proof. Let us write $f(s,u) =: f = \sum_{0<i}f_it^i$. The expression $id\ell_{j-i}\wedge\ell_i$ evaluated on $\delta(s\exp(u))$ is equal to

$$id(f_{j-i})u_i - if_idu_{j-i} = id(f_{j-i})u_i + (j-i)f_idu_{j-i} - jf_idu_{j-i}.$$

Summing these, we find that

$$\alpha_j(\delta(s\exp(u))) = \sum_{1\leq i\leq j-1}(id(f_{j-i})u_i + (j-i)f_idu_{j-i}) - j\sum_{1\leq i\leq j-1}f_idu_{j-i}.$$

Let $Du := \sum_{1\leq i}du_it^i$ and $u_t := \frac{\partial u}{\partial t}$. Then the last expression can be rewritten as

$$t_{j-1}(D(fu_t)) - jt_j(fDu) = t_{j-1}(D(fu_t) - (fDu)_t) = t_{j-1}(Dfu_t - f_tDu). \tag{5.1.2}$$

We would like to see that the coefficient of du_i in (5.1.2) is equal to 0. The coefficient of du_i in $Df = D\log(\frac{1-s\exp(u)}{1-s})$ is equal to $\frac{-s\exp(u)}{1-s\exp(u)}t^i$. Therefore, the coefficient of du_i in (5.1.2) is

$$t_{j-1-i}\Big(\frac{-s\exp(u)}{1-s\exp(u)}u_t - f_t\Big).$$

Since $f_t = \frac{\partial}{\partial t}(\log(1-s\exp(u))) = \frac{-s\exp(u)}{1-s\exp(u)}u_t$, the last expression is 0.

Therefore $\alpha_j(\delta(s\exp(u)))$ does not depend on the du_i's, and we can rewrite (5.1.2) as

$$\alpha_j(\delta(s\exp(u))) = t_{j-1}(Dfu_t - f_tDu) = t_{j-1}(f_su_t)ds,$$

where $f_s = \frac{\partial f}{\partial s}$. □

Lemma 5.1.2. *If $u = u|_m$ and $m \le j < r$, we have*

$$jt_j(f) = st_{j-1}(f_su_t).$$

Proof. The expression $jt_j(f) - st_{j-1}(f_su_t)$ is equal to

$$t_{j-1}(f_t - s(f_su_t)) = t_{j-1}(\frac{\partial}{\partial t}\log(1-s\exp(u)) - s\frac{\partial}{\partial s}\log(\frac{1-s\exp(u)}{1-s})\cdot u_t).$$

Since

$$\frac{\partial}{\partial t}\log(1-s\exp(u)) = \frac{-s\exp(u)}{1-s\exp(u)}\cdot u_t = s\frac{\partial}{\partial s}\log(1-s\exp(u))\cdot u_t,$$

the above expression is equal to $t_{j-1}(\frac{s}{s-1}u_t)$, which is 0, under the assumption that $u = u_1t + \cdots + u_{m-1}t^{m-1}$ and $m \le j$. □

Let $d\ell_0 : R_\infty^\times \to \Omega_R^1$ be defined as $d\ell_0(\alpha) := d\log(\alpha(0))$. Note that ℓ_0 itself is not defined, even though ℓ_i are defined for $i > 0$.

Proposition 5.1.3. *The map $M_{m,r}$ defined as*

$$M_{m,r} := \ell i_{m,r}\otimes d\ell_0 - \sum_{m\le j<r}\frac{r-j}{j}(\alpha_j\circ\delta)\otimes \ell_{r-j}$$

gives a map from $B_2(R_\infty)\otimes R_\infty^\times$ to Ω_R^1, of $\star$-weight r, which vanishes on the image under δ of those $[se^u]\in\mathbb{Q}[R_\infty^\flat]$, with $u = u|_m$. The map $M_{m,m+1}$ descends to a map from $B_2(R_m)\otimes R_m^\times$.

Proof. That $M_{m,r}$ is of $\star$-weight r follows immediately from the expression for $\alpha_j(\delta(s\exp(u)))$ in Lemma 5.1.1, which shows that $\alpha_j(\delta(s\exp(u)))$ is of $\star$-weight j.

Let us now show that $M_{m,r}$ evaluated on $[s\exp(u)]\otimes s\exp(u)$, with $u = u|_m$, is equal to 0. By Lemma 5.1.1, $\alpha_j(\delta(s\exp(u)))$ is equal to $t_{j-1}(f_su_t)ds$. This implies that $\sum_{m\le j<r}\frac{r-j}{j}(\alpha_j\circ\delta)\otimes\ell_{r-j}$ evaluated on $[s\exp(u)]\otimes s\exp(u)$ is equal to

$$\sum_{m\le j<r}\frac{r-j}{j}t_{j-1}(f_su_t)u_{r-j}ds = \frac{1}{s}\sum_{m\le j<r}(r-j)t_j(f)u_{r-j}ds,$$

by Lemma 5.1.2, since $u = u|_m$. The final expression can be rewritten as

$$t_{r-1}(f\cdot u_t|_{r-m})\frac{ds}{s} = \ell i_{m,r}(s\exp(u))\frac{ds}{s} = (\ell i_{m,r}\otimes d\ell_0)([s\exp(u)]\otimes s\exp(u))$$

since $u = u|_m$. This proves the first part of the proposition.

When $r = m+1$, $M_{m,r}$ takes the form

$$\ell i_{m,m+1}\otimes d\ell_0 - \frac{1}{m}\big((d\ell_{m-1}\wedge\ell_1 + 2d\ell_{m-2}\wedge\ell_2 + \cdots + (m-1)d\ell_1\wedge\ell_{m-1})\circ\delta\big)\otimes\ell_1.$$

Since all the functions in this expression depend on the classes of the elements in R_m, the statement easily follows. □

5.2. **The regulator maps from $\mathrm{H}^2(R_m,\mathbb{Q}(3))$ to Ω^1_R.** We would like to define maps

$$L_{m,r}:\mathrm{H}^2(R_m,\mathbb{Q}(3))\to\Omega^1_R$$

based on the maps $M_{m,r}$ in Proposition 5.1.3. The problem with $M_{m,r}$ is that it does not descend to a map on $B_2(R_m)\otimes R_m^\times$, if $r\neq m+1$. We will modify $M_{m,r}$ slightly to correct this defect but keep the other properties to obtain $L_{m,r}$.

In order to simplify the notation from now on we are going to let $\ell i_{m,m}:=0$. Note that $\ell i_{m,r}$ was previously defined only when $m+1\leq r\leq 2m-1$ so this will not cause any confusion. We define $\beta_m(j)$, for $m\leq j<2m-1$, by

$$\begin{aligned}\beta_m(j) &:= d\ell i_{m,j}+\sum_{\substack{a+b=j\\1\leq a,b<m}} b(d\ell_a\wedge\ell_b)\circ\delta=\sum_{\substack{a+b=j\\m\leq a,\,1\leq b}} bd(\ell_a\wedge\ell_b)\circ\delta+\sum_{\substack{a+b=j\\1\leq a,b<m}} b(d\ell_a\wedge\ell_b)\circ\delta\\ &= d\ell i_{m,j}+\alpha_j\circ\delta-\sum_{1\leq a\leq j-m}\big((j-a)d\ell_a\wedge\ell_{j-a}+ad\ell_{j-a}\wedge\ell_a\big)\circ\delta\end{aligned}$$

and

$$L_{m,r}:=\ell i_{m,r}\otimes d\ell_0-\sum_{m\leq j<r}\big(\frac{r-j}{j}\beta_m(j)\otimes\ell_{r-j}-\ell i_{m,j}\otimes d\ell_{r-j}\big). \tag{5.2.1}$$

We would like to emphasize that, because of our conventions, the summand that corresponds to $j=m$ is equal to $\frac{r-m}{m}\alpha_m\otimes\ell_{r-m}$ exactly as in the case of $M_{m,r}$, the terms corresponding to $m<j$ are modified however.

Lemma 5.2.1. *With the above definition, $L_{m,r}$ defines a map from $B_2(R_m)\otimes R_m^\times$ to Ω^1_R of $\star$-weight r.*

Proof. Since all the terms in the definition of $L_{m,r}$ depend on the variables modulo t^m, we obtain a map from $B_2(R_m)\otimes R_m^\times$ to Ω^1_R.

Since we know that $M_{m,r}$ is of $\star$-weight r, in order to prove that $L_{m,r}$ is of $\star$-weight r, it suffices to prove the same for $L_{m,r}-M_{m,r}$. This difference is equal to the sum of

$$-\sum_{m\leq j<r}\big(\frac{r-j}{j}d\ell i_{m,j}\otimes\ell_{r-j}-\ell i_{m,j}\otimes d\ell_{r-j}\big) \tag{5.2.2}$$

and

$$\sum_{m\leq j<r}\frac{r-j}{j}\big(\sum_{1\leq a\leq j-m}\big((j-a)d\ell_a\wedge\ell_{j-a}+ad\ell_{j-a}\wedge\ell_a\big)\big)\circ\delta\otimes\ell_{r-j}. \tag{5.2.3}$$

Let us first look at the term $(j-a)d\ell_a\wedge\ell_{j-a}+ad\ell_{j-a}\wedge\ell_a$. For any $u\wedge v\in\Lambda^2 tR_\infty$ and $\lambda\in R^\times$, $((j-a)d\ell_a\wedge\ell_{j-a}+ad\ell_{j-a}\wedge\ell_a)(\lambda\star(u\wedge v))=$

$$\begin{aligned}&\big((j-a)d(\lambda^a\ell_a)\wedge\lambda^{j-a}\ell_{j-a}+ad(\lambda^{j-a}\ell_{j-a})\wedge\lambda^a\ell_a\big)((u\wedge v))\\ =\;&\big(\lambda^j((j-a)d\ell_a\wedge\ell_{j-a}+ad\ell_{j-a}\wedge\ell_a)+(j-a)a(\ell_a\wedge\ell_{j-a}+\ell_{j-a}\wedge\ell_a)\lambda^{j-1}d\lambda\big)(u\wedge v)\\ =\;&\lambda^j((j-a)d\ell_a\wedge\ell_{j-a}+ad\ell_{j-a}\wedge\ell_a)(u\wedge v).\end{aligned}$$

Therefore the term (5.2.3) is of $\star$-weight r.

Similarly, $(r-j)d\ell i_{m,j}\otimes\ell_{r-j}-j\ell i_{m,j}\otimes d\ell_{r-j}$ evaluated on $\lambda\star(u\otimes v)$ is equal to

$$\begin{aligned}&(r-j)d(\lambda^j\ell i_{m,j})\otimes\lambda^{r-j}\ell_{r-j}-j\lambda^j\ell i_{m,j}\otimes d(\lambda^{r-j}\ell_{r-j})\\ =\;&\lambda^r\big((r-j)d\ell i_{m,j}\otimes\ell_{r-j}-j\ell i_{m,j}\otimes d\ell_{r-j}\big)+(r-j)j(\ell i_{m,j}\otimes\ell_{r-j}-\ell i_{m,j}\otimes\ell_{r-j})\lambda^{r-1}d\lambda\\ =\;&\lambda^r\big((r-j)d\ell i_{m,j}\otimes\ell_{r-j}-j\ell i_{m,j}\otimes d\ell_{r-j}\big)\end{aligned}$$

evaluated on $u\otimes v$. This implies that the term (5.2.2) is of $\star$-weight r and finishes the proof of the lemma. □

Proposition 5.2.2. *The map $L_{m,r}:B_2(R_m)\otimes R_m^\times\to\Omega^1_R$ vanishes on the boundaries of the elements in $\mathbb{Q}[R_m^\flat]$ and hence induces a map*

$$(B_2(R_m)\otimes R_m^\times)/im(\delta)\to\Omega^1_R,$$

which by restriction gives the regulator map $\mathrm{H}^2(R_m, \mathbb{Q}(3)) \to \Omega^1_R$ *of* $\star$*-weight* r *we were looking for. We continue to denote these two induced maps by the same notation* $L_{m,r}$.

Proof. We know that $M_{m,r}$ vanishes on the boundary $\delta(s\exp(u))$ of elements $s\exp(u) \in \mathbb{Q}[R_\infty]$, with $u = u|_m$, by Proposition 5.1.3. We also know by the previous lemma that $L_{m,r}$ descends to a map on $B_2(R_m) \otimes R_m^\times$. Therefore, in order to prove the statement, we only need to prove that

$$L_{m,r}(\delta(s\exp(u))) = M_{m,r}(\delta(s\exp(u))),$$

for $u = u|_m$. We first rewrite $L_{m,r}$ as the composition of $\delta \otimes id$ with

$$\sum_{\substack{a+b=r\\ m\le a,\,1\le b}} b\cdot \ell_a \wedge \ell_b \otimes d\ell_0 - \sum_{\substack{a+b+c=r\\ m\le a+b,\\ 1\le a,\,b,\,c<m}} \frac{c}{a+b} b\cdot d\ell_a \wedge \ell_b \otimes \ell_c$$
$$- \sum_{\substack{a+b+c=r\\ m\le a,\,1\le b,\,c}} \frac{c}{a+b} b\cdot d(\ell_a \wedge \ell_b) \otimes \ell_c + \sum_{\substack{a+b+c=r\\ m\le a,\,1\le b,\,c}} b\cdot \ell_a \wedge \ell_b \otimes d\ell_c.$$

On the other hand, recall that $M_{m,r}$ is the composition of $\delta \otimes id$ with

$$\sum_{\substack{a+b=r\\ m\le a,\,1\le b}} b\cdot \ell_a \wedge \ell_b \otimes d\ell_0 - \sum_{\substack{a+b+c=r\\ m\le a+b,\\ 1\le a,\,b,\,c}} \frac{c}{a+b} b\cdot d\ell_a \wedge \ell_b \otimes \ell_c.$$

If we compare the two expressions we see that all of the terms match above except possibly the ones that correspond to the triples (a, b, c) with $1 \le a,\, b,\, c,\, a+b+c = r$, and $m \le a$ or $m \le b$. By anti-symmetry, we may assume without loss of generality that $m \le a$. We need to compare the coefficients of the terms $d\ell_a \wedge \ell_b \otimes \ell_c$, $\ell_a \wedge d\ell_b \otimes \ell_c$, and $\ell_a \wedge \ell_b \otimes d\ell_c$, subject to the above constraints, in $L_{m,r}$ and $M_{m,r}$.

The coefficient of $d\ell_a \wedge \ell_b \otimes \ell_c$ in $L_{m,r}$ and $M_{m,r}$ are both equal to $-\frac{cb}{a+b}$. The coefficient of $\ell_a \wedge d\ell_b \otimes \ell_c$ in $L_{m,r}$ is $\frac{-cb}{a+b}$ and in $M_{m,r}$, it is $\frac{ca}{a+b}$. Finally, the coefficient of $\ell_a \wedge \ell_b \otimes d\ell_c$ in $L_{m,r}$ is b, whereas in $M_{m,r}$ it is 0.

We finally note that the values of $\ell_a \wedge d\ell_b \otimes \ell_c$ and $\ell_a \wedge \ell_c \otimes d\ell_b$ on $\delta(s\exp(u)) \otimes s\exp(u)$ are the same when $u = u|_m$. Then the equality $\frac{-cb}{a+b} + c = \frac{ca}{a+b}$ finishes the proof. □

We can restate Lemma 5.2.1 as follows. First, let

$$\gamma_m(j) := d\ell_{m,j} + \sum_{\substack{a+b=j\\ 1\le a,b<m}} b(d\ell_a \wedge \ell_b).$$

Note that since $\ell i_{m,j} = \ell_{m,j} \circ \delta$ by Definition 3.0.2, we have $\beta_m(j) = \gamma_m(j) \circ \delta$. Finally, if we let

$$N_{m,r} := \ell_{m,r} \otimes d\ell_0 - \sum_{m\le j<r} \big(\frac{r-j}{j}\gamma_m(j) \otimes \ell_{r-j} - \ell_{m,j} \otimes d\ell_{r-j}\big),$$

then by (5.2.1), we have the following.

Corollary 5.2.3. *For* $2 \le m < r < 2m$, *we have a commutative diagram*

$$\begin{array}{ccc} B_2(R_r) \otimes R_r^\times & \xrightarrow{\delta\otimes id} & \Lambda^2 R_r^\times \otimes R_r^\times \\ \downarrow & & \downarrow N_{m,r} \\ B_2(R_m) \otimes R_m^\times & \xrightarrow{L_{m,r}} & \Omega^1_R. \end{array}$$

We expect that the above maps combine to give an isomorphism between the infinitesimal part of the cohomology of R_m and the direct sum of the module of Kähler differentials, justifying the name of the regulator, cf. [6, Conjecture 1.15]. However, at this point, we can only prove the surjectivity:

Proposition 5.2.4. *Suppose that* R *is a regular local* $\mathbb{Q}$*-algebra and* $2 \le m$ *as the above. The direct sum of the* $L_{m,r}$ *induces a surjection:*

$$\oplus_{m<r<2m} L_{m,r} : \mathrm{H}^2(R_m, \mathbb{Q}(3))^\circ \twoheadrightarrow \oplus_{m<r<2m}\Omega^1_R.$$

Proof. Suppose that $\alpha \in B_2(R_m)^\circ$ is in the part of kernel of the δ° which is of $\star$-weight r. By Theorem 3.0.3, this part is isomorphic to R via the restriction of the map $\ell i_{m,r}$. Computing the value of $L_{m,r}$ on $\alpha \otimes b$, for $b \in R^\times$, we see that $L_{m,r}(\alpha \otimes b) = \ell i_{m,r}(\alpha)\frac{db}{b}$. Since $\alpha \otimes b$ is in the kernel of δ, we see that the image of $L_{m,r}$ above is the additive group generated by the set $Rd\log(R^\times)$. Since R is local this is equal to Ω^1_R. This implies the surjectivity. $\square$

Conjecture 5.2.5. *We conjecture that the map $\oplus_{m<r<2m}L_{m,r}$ in Proposition 5.2.4 is injective and hence is an isomorphism.*

6. Construction of the maps from $\Lambda^3(R_{2m-1},(t^m))^\times$ to $\Omega^1_{R/k}$

For a ring A and ideal I, let $(A,I)^\times := \{(a,b)|a,\, b \in A^\times,\ a-b \in I\}$, and let $\pi_i : (A,I)^\times \to A^\times$, for $i = 1,2$ denote the two projections. If R is a k-algebra, in this section we will define a map $\omega_{m,r} : \Lambda^3(R_r,(t^m))^\times \to \Omega^1_{R/k}$.

6.1. **Definition of $\Omega_{m,r}$.** Assume that R is $\mathbb{Q}$-algebra and $2 \le m < r < 2m$. Let us put $I_{m,r} := im\big((1+(t^m)) \otimes \Lambda^2 R_r^\times\big) \subseteq (\Lambda^3 R_r^\times)^\circ$.

Definition 6.1.1. *We define the map $\Omega_{m,r} : I_{m,r} \to \Omega_R$ by the following formulae:*

(i) if $x \ge m;\ x+y+z = r;\ \ y,\, z \ge 1;$ and $a,\, b,\, c \in R$ then:

$$\Omega_{m,r}(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)) := a(yb \cdot dc - zc \cdot db);$$

(ii) if $x \ge m;\ x+y = r;\ y \ge 1;\ a,\, b \in R;$ and $\gamma \in R^\times$ then:

$$\Omega_{m,r}(\exp(at^x) \wedge \exp(bt^y) \wedge \gamma) := a(yb \cdot \frac{d\gamma}{\gamma});$$

(iii) if $x \ge m;\ x+z = r;\ z \ge 1;\ a,\, c \in R;$ and $\beta \in R^\times$ then:

$$\Omega_{m,r}(\exp(at^x) \wedge \beta \wedge \exp(ct^z)) := a(-zc \cdot \frac{d\beta}{\beta});$$

(iv) if $x = r;$ and $\beta,\, \gamma \in R^\times$ then:

$$\Omega_{m,r}(\exp(at^x) \wedge \beta \wedge \gamma) := 0.$$

Remark 6.1.2. Notice that in case (ii) of the above definition, using the notation $\exp(ct^z)$ with $z = 0$ instead of γ would not make sense. This is because in order for $\exp(ct^z)$ to be well-defined, we need $ct^z \in (t) \subseteq R_r$. However, if we continue to use this notation $\exp(c)$, without specifying what c is and without the notation making actual sense, we note that the formula (ii) becomes a special case of formula (i) in the following sense. If we formally put $\gamma = \exp(c)$ then again formally $\log(\gamma) = c$ and $\frac{d\gamma}{\gamma} = d\log(\gamma) = dc$. This makes formula (ii) exactly the same as formula (i) if we also note that since we put $z = 0$ the term involving $zc.db$ disappears in (i). We will be using these notations and conventions in order to shorten the expressions in the remaining of the paper. However, we would like to emphasize that when proving the statements under consideration we are always using the Definition 6.1.1 since these notations are only formal and do not make actual sense. Similar comments apply to (iii) when $y = 0$ and to (iv) when $y = z = 0$.

To sum up we will write that, if $x \ge m;\ x+y+z = r;$ and $y, z \ge 0$:

$$\Omega_{m,r}(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)) = a(yb \cdot dc - zc \cdot db). \tag{6.1.1}$$

In the proof of the next proposition, we will use the following notation. Recall that $\mathbb{Q}_r = \mathbb{Q}[t]/(t^r)$. Since we assume that R contains $\mathbb{Q}$, R_r is a $\mathbb{Q}_r$-algebra. Let $\underline{d} : R_r \to \Omega^1_{R_r/\mathbb{Q}_r}$ denote the canonical differential. Note that $\underline{d}$ has the property that $\underline{d}(t) = 0$. There is a natural isomorphism

$$\oplus_{0 \le i < r} t^i \Omega^1_R \to \Omega^1_{R_r/\mathbb{Q}_r}.$$

Proposition 6.1.3. *Suppose that $\hat{f},\, \tilde{f} \in R_r^\flat$ and $\hat{g},\, \tilde{g} \in R_r^\times$ have the same reductions modulo (t^m), with $2 \le m < r < 2m$. Then we have*

$$\Omega_{m,r}(\delta([\hat{f}]) \wedge \hat{g} - \delta([\tilde{f}]) \wedge \tilde{g}) = 0$$

and

$$\Omega_{m,r}(\delta([\hat{f}])\wedge\hat{g}-\delta([\tilde{f}])\wedge\hat{g})=0.$$

Proof. By the assumptions $\hat{g}/\tilde{g}$ is a product of terms of the form $\exp(at^x)$ with $a\in R$ and $m\le x<r$. Hence, in order to prove the first equality we need to prove that $\Omega_{m,r}$ vanishes on $\delta([\hat{f}])\wedge\exp(at^x)$. By the definition of $\Omega_{m,r}$ above, we have

$$\Omega_{m,r}(\delta([\hat{f}])\wedge\exp(at^x))=-a\cdot res_{t=0}\frac{1}{t^{r-x}}(d\log\wedge d\log)((1-\hat{f})\wedge\hat{f})=0,$$

since $(d\log\wedge d\log)((1-\hat{f})\wedge\hat{f})=0$.

In order to prove the second equality, note that $\hat{g}$ is a product of terms of type $\exp(ct^z)$ with $0\le z$, where for $z=0$, we use the notation in Remark 6.1.2. Then using the first equality, we only need to prove that

$$\Omega_{m,r}(\delta([\hat{f}])\wedge\exp(ct^z)-\delta([\tilde{f}])\wedge\exp(ct^z))=0, \tag{6.1.2}$$

for $0\le z<m$. On the other hand, since $\hat{f}$ and $\tilde{f}$ are equal modulo (t^m), we see that (6.1.2) holds when $r-z<m$. Therefore from now on we assume that $m\le r-z$.

If we knew (6.1.2) in the special case when $\hat{f}=\tilde{f}+at^x$ with $m\le x$, then by successively using this information we obtain (6.1.2) for any $\hat{f}$ and $\tilde{f}$ which have the same reduction modulo (t^m). Therefore, from now on we assume that $\hat{f}=\tilde{f}+at^x$, and $\tilde{f}=s+b_1t+b_2t^2+\cdots$.

The left hand side of (6.1.2)) is a sum of two terms: one containing the term dc and the other one containing the term c.

Let us first consider the term containing dc. The term containing dc is 0 when $r-z=m$ since $m\le x$. Therefore we assume that $m<r-z$. In this case, we compute that this term containing dc is equal to

$$\Big(\big(\sum_{1\le i\le r-z-m}i\cdot\ell_{r-z-i}\wedge\ell_i\big)\circ\delta\Big)([\hat{f}]-[\tilde{f}])\cdot dc. \tag{6.1.3}$$

We have $\Big(\big(\sum_{1\le i\le r-z-m}i\cdot\ell_{r-z-i}\wedge\ell_i\big)\circ\delta\Big)([h])=\ell i_{m,r-z}([h|_{t^m}])$, by the definition of $\ell i_{m,r-z}$, for any $h\in R_r^\flat$. This implies that (6.1.3) is equal to $(\ell i_{m,r-z}([\hat{f}|_{t^m}])-\ell i_{m,r-z}([\tilde{f}|_{t^m}]))\cdot dc=0$.

Finally, we consider summand of (6.1.2) which contain the term c. Letting $\beta:=b_1t+b_2t^2+\cdots$, this term is equal to $-z\cdot c\cdot a$ times the coefficient of t^{r-z-x} in

$$\sum_{0<i,j}(-1)^{i-1}\frac{\beta^{i-1}}{(s-1)^i}\cdot\underline{d}\big(\frac{(-1)^{j-1}}{j}\frac{\beta^j}{s^j}\big)-\Big((-1)^{i-1}\frac{\beta^{i-1}}{s^i}\cdot\underline{d}\big(\frac{(-1)^{j-1}}{j}\frac{\beta^j}{(s-1)^j}\big)\Big)$$
$$+\sum_{0<i}(-1)^{i-1}\frac{\beta^{i-1}}{(s-1)^i}\cdot\frac{ds}{s}-\big((-1)^{i-1}\frac{\beta^{i-1}}{s^i}\cdot\frac{ds}{s-1}\big).$$

Let us rewrite the last expression as

$$\sum_{0<i,j}(-1)^{i+j}\beta^{i+j-2}\underline{d}(\beta)\big(\frac{1}{(s-1)^is^j}-\frac{1}{s^i(s-1)^j}\big)$$
$$+\sum_{0<i,j}(-1)^{i+j-1}\beta^{i+j-1}ds\big(\frac{1}{(s-1)^is^{j+1}}-\frac{1}{s^i(s-1)^{j+1}}\big)$$
$$+\sum_{0<i}(-1)^{i-1}\beta^{i-1}ds\big(\frac{1}{(s-1)^is}-\frac{1}{s^i(s-1)}\big).$$

In this expression, the first sum is equal to 0. In the second sum only the terms with $i=1$ survive, to make the sum equal to

$$\sum_{0<j}(-1)^j\beta^jds\big(\frac{1}{(s-1)s^{j+1}}-\frac{1}{s(s-1)^{j+1}}\big).$$

This is precisely the negative of the third sum above. This finishes the proof. □

Let $s : \Lambda^3(R_r, (t^m))^\times \to \Lambda^3 R_r^\times$, be given by $s := \Lambda^3\pi_1 - \Lambda^3\pi_2$. The group $\Lambda^3(R_r, (t^m))^\times$ is generated by elements of the form $(a, a') \wedge (b, b') \wedge (c, c')$ with $a - a'$, $b - b'$, $c - c' \in (t^m)$ and a, a', b, b', c, $c' \in R_r^\times$. If we let $a = \alpha a'$, $b = \beta b'$, and $c = \gamma c'$, then α, β, $\gamma \in 1 + (t^m)$. We have

$$s((a, a') \wedge (b, b') \wedge (c, c')) = \alpha a' \wedge \beta b' \wedge \gamma c' - a' \wedge b' \wedge c' \in I_{m,r},$$

since the last expression is a sum of elements of the form $\delta \wedge d \wedge e$ with $\delta \in 1 + (t^m)$ and d, $e \in R_r^\times$. This implies that s factors through $I_{m,r} \subseteq (\Lambda^3 R_r^\times)^\circ$.

Definition 6.1.4. *Suppose that R is a k-algebra. We let $\underline{\Omega}_{m,r}$ denote the composition of $\Omega_{m,r}$ with the canonical projection $\Omega^1_R \to \Omega^1_{R/k}$. We define $\omega_{m,r} : \Lambda^3(R_r, (t^m))^\times \to \Omega^1_{R/k}$ as the composition $\underline{\Omega}_{m,r} \circ s$ of $s : \Lambda^3(R_r, (t^m))^\times \to I_{m,r}$, and $\underline{\Omega}_{m,r} : I_{m,r} \to \Omega^1_{R/k}$.*

6.2. **Relation of $\underline{\Omega}_{m,r}$ to $\underline{L}_{m,r}$.** In this section, we assume that R is a smooth local k-algebra of relative dimension 1. We will relate the construction $\Omega_{m,r}$ to $L_{m,r}$, assuming Conjecture 5.2.5. Even though the results in this section will not be used in the rest of the paper, we include this section since it gives a more conceptual description of $\Omega_{m,r}$ and it will be referred to in future work. Let us denote the composition of $L_{m,r}$ with the canonical projection $\Omega^1_R \to \Omega^1_{R/k}$ by $\underline{L}_{m,r}$.

Given $\alpha \in I_{m,r}$, we will show that there exists

$$\varepsilon \in im((1 + (t^m)) \otimes k^\times \otimes R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ,$$

such that $\alpha - \varepsilon = \delta_r(\gamma)$ for some $\gamma \in (B_2(R_r) \otimes R_r^\times)^\circ$. Assuming Conjecture 5.2.5, we will then show that

$$\underline{\Omega}_{m,r}(\alpha) = \underline{L}_{m,r}(\gamma|_{t^m}) \in \Omega^1_{R/k}.$$

Lemma 6.2.1. *For any $\alpha \in (\Lambda^3 R_r^\times)^\circ$, there exists $\varepsilon \in im(\Lambda^2 R_r^\times \otimes k^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$ such that $\alpha - \varepsilon$ lies in the image of $(B_2(R_r) \otimes R_r^\times)^\circ$ in $(\Lambda^3 R_r^\times)^\circ$. Moreover, if*

$$\alpha \in im((1 + t^m R_r)^\times \otimes \Lambda^2 R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$$

then we can choose

$$\varepsilon \in im((1 + t^m R_r)^\times \otimes R_r^\times \otimes k^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$$

such that $\alpha - \varepsilon$ lies in the image of $(B_2(R_r) \otimes R_r^\times)^\circ$ in $(\Lambda^3 R_r^\times)^\circ$.

Proof. The infinitesimal part of the cokernel of the map

$$B_2(R_r) \otimes R_r^\times \to \Lambda^3 R_r^\times$$

is $K_3^M(R_r)^\circ$ which is isomorphic to $\oplus_{1 \le i < r} t^i \Omega^2_R$ via the map from $\Lambda^3 R_r^\times$, whose i-th coordinate is given by

$$res_{t=0} \frac{1}{t^i} d\log(y_1) \wedge d\log(y_2) \wedge d\log(y_3), \tag{6.2.1}$$

by Corollary 4.0.7. Further, by the assumption on smoothness of dimension 1, we conclude that the natural map

$$\Omega^1_R \otimes_k \Omega^1_k \to \Omega^2_R$$

is surjective. Since we assume that R is local, the map

$$R \otimes_{\mathbb{Z}} R^\times \to \Omega^1_R,$$

which sends $a \otimes b$ to $a \cdot d\log b$, is surjective.

Note that the image of $\exp(t^i u) \wedge v \wedge \lambda$, with $u \in R$, $v \in R^\times$ and $\lambda \in k^\times$ under the i-th map in (6.2.1) is $i \cdot u \cdot d\log(v) \wedge d\log(\lambda)$ and under the coordinate j maps in (6.2.1) with $j \ne i$, the image is 0. Together with the above, this shows that the map

$$(\Lambda^2 R_r^\times \otimes k^\times)^\circ \to (\Lambda^3 R_r^\times)^\circ \to \oplus_{1 \le i < r} t^i \Omega^2_R$$

is surjective and hence proves the first statement.

For the second statement, note that if $\alpha \in im((1+t^m R_r)^\times \otimes \Lambda^2 R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$ then its image in $\oplus_{1 \le i < r} t^i \Omega^2_R$ lands in the summand $\oplus_{m \le i < r} t^i \Omega^2_R$. Since by the above discussion, we also see that the composition

$$(1 + t^m R_r)^\times \otimes R_r^\times \otimes k^\times \to (\Lambda^3 R_r^\times)^\circ \to \oplus_{m \le i < r} t^i \Omega^2_R$$

is surjective, the second statement similarly follows. □

The next lemma is crucial in relating $\underline{\Omega}_{m,r}$ to $\underline{L}_{m,r}$.

Lemma 6.2.2. *Let $A(r)$ denote the kernel of the differential $B_2(R_r) \otimes R_r^\times \to \Lambda^3 R_r^\times$. Then Conjecture 5.2.5 implies that the composition*

$$A(r) \subseteq B_2(R_r) \otimes R_r^\times \xrightarrow{|_{t^m}} B_2(R_m) \otimes R_m^\times \xrightarrow{L_{m,r}} \Omega^1_R$$

is 0.

Proof. By Proposition 5.2.2, we know that $L_{m,r}$ induces a map from $\mathrm{H}^2(R_m, \mathbb{Q}(3))$ to Ω^1_R of weight r. The composition of the maps in the statement of the lemma can be rewritten as the composition

$$A(r) \longrightarrow \mathrm{H}^2(R_r, \mathbb{Q}(3)) \xrightarrow{|_{t^m}} \mathrm{H}^2(R_m, \mathbb{Q}(3)) \xrightarrow{L_{m,r}} \Omega^1_R.$$

By Conjecture 5.2.5 the $\star$-weights of $\mathrm{H}^2(R_r, \mathbb{Q}(3))$ are between $r+1$ and $2r-1$. This implies that the map from $\mathrm{H}^2(R_r, \mathbb{Q}(3)) \to \Omega^1_R$, which is of weight r, is in fact the zero map and finishes the proof. □

Remark 6.2.3. We emphasize that, in the above lemma, we prove that $L_{m,r}(\gamma|_{t^m}) = 0$ for $\gamma \in A(r)$. If γ is only assumed to be in the kernel of the map $B_2(R_m) \otimes R_m^\times \to \Lambda^3 R_m^\times$ then $L_{m,r}(\gamma)$ need not be equal to 0.

Lemma 6.2.4. *Assuming Conjecture 5.2.5, suppose that $\gamma \in B_2(R_r) \otimes R_r^\times$ such that*

$$\delta_r(\gamma) \in im(\big(\oplus_{0 \leq s < r} (1 + t^s R_r)^\times \otimes (1 + t^{r-s} R_r)^\times\big) \otimes k^\times) \subseteq \Lambda^3 R_r^\times$$

then $\underline{L}_{m,r}(\gamma|_{t^m}) = 0$.

Proof. First let $\alpha := \exp(ut^i) \wedge \exp(vt^j) \wedge \lambda \in \Lambda^3 R_r^\times$ with $u, v \in R$ and $\lambda \in k^\times$ and $r \leq i + j$. We have $\exp(ut^i) \wedge \exp(vt^j) \in (\Lambda^2 R_r^\times)^\circ$ and

$$res_t \frac{1}{t^a} d\log(\exp(ut^i)) \wedge d\log(\exp(vt^j)) = res_t \frac{1}{t^a} d(ut^i) \wedge d(vt^j) = 0 \in \Omega^1_R,$$

for all $1 \leq a < r$, since $r \leq i + j$. This implies that there is $\alpha_0 \in B_2(R_r)$ such that $\delta_r(\alpha_0) = \exp(ut^i) \wedge \exp(vt^j)$, and hence $\delta_r(\alpha_0 \otimes \lambda) = \alpha$.

Let us compute $L_{m,r}((\alpha_0 \otimes \lambda)|_{t^m})$. Since $\ell_i(\lambda) = 0$, for $0 < i$ by the formula for $L_{m,r}$ we see that

$$L_{m,r}((\alpha_0 \otimes \lambda)|_{t^m}) = \ell i_{m,r}(\alpha_0|_{t^m}) \cdot d\log(\lambda) \in \Omega^1_R.$$

This expression vanishes in $\Omega^1_{R/k}$ and therefore $\underline{L}_{m,r}((\alpha_0 \otimes \lambda)|_{t^m}) = 0$. Taking the sum of expressions such as above, we deduce that if

$$\alpha \in im(\big(\oplus_{0 \leq s < r} (1 + t^s R_r)^\times \otimes (1 + t^{r-s} R_r)^\times\big) \otimes k^\times) \subseteq \Lambda^3 R_r^\times$$

then there is a $\tilde{\alpha} \in B_2(R_r) \otimes R_r^\times$ such that $\delta_r(\tilde{\alpha}) = \alpha$ and $\underline{L}_{m,r}(\tilde{\alpha}|_{t^m}) = 0$.

Applying this to $\alpha := \delta_r(\gamma)$ we deduce that there exists $\tilde{\alpha} \in B_2(R_r) \otimes R_r^\times$ such that $\delta_r(\tilde{\alpha}) = \delta_r(\gamma)$ and $\underline{L}_{m,r}(\tilde{\alpha}|_{t^m}) = 0$. Then we have

$$\underline{L}_{m,r}(\gamma|_{t^m}) = \underline{L}_{m,r}(\tilde{\alpha}|_{t^m}) + \underline{L}_{m,r}((\gamma - \tilde{\alpha})|_{t^m}) = \underline{L}_{m,r}((\gamma - \tilde{\alpha})|_{t^m}).$$

Since $\delta_r(\gamma - \tilde{\alpha}) = 0$, the last expression is 0 by Lemma 6.2.2. □

Lemma 6.2.5. *Assuming Conjecture 5.2.5, if $\gamma \in B_2(R_r) \otimes R_r^\times$ such that*

$$\delta_r(\gamma) \in im((\Lambda^2 R_r^\times)^\circ \otimes k^\times) \subseteq \Lambda^3 R_r^\times$$

then $\underline{L}_{m,r}(\gamma|_{t^m}) = 0$.

Proof. Suppose that γ is as in the statement of the lemma. Fix some $a \in \mathbb{Z}_{>1}$. We inductively define $\gamma^{[i]}$ as follows. Let $\gamma^{[-1]} = \gamma$, and

$$\gamma^{[i]} := a \star \gamma^{[i-1]} - a^i \gamma^{[i-1]},$$

for $0 \le i < r$. Since $L_{m,r}$ is of weight r, $L_{m,r}(\gamma^{[i]}|_{t^m}) = (a^r - a^i)L_{m,r}(\gamma^{[i-1]}|_{t^m})$. Therefore proving that $\underline{L}_{m,r}(\gamma|_{t^m}) = 0$ is equivalent to proving that $\underline{L}_{m,r}(\gamma^{[r-1]}|_{t^m}) = 0$. On the other hand,

$$\delta(\gamma^{[r-1]}) \subseteq im(\big(\oplus_{0\le s<r} (1+t^sR_r)^\times \otimes (1+t^{r-s}R_r)^\times\big) \otimes k^\times),$$

and therefore the previous lemma implies that $\underline{L}_{m,r}(\gamma^{[r-1]}|_{t^m}) = 0$. □

Assuming Conjecture 5.2.5, we construct a map $\underline{\tilde{\Omega}}_{m,r} : I_{m,r} \to \Omega_{R/k}$, using $\underline{L}_{m,r}$ as follows. Starting with $\alpha \in I_{m,r}$, we know, by Lemma 6.2.1, that there exists

$$\varepsilon \in im((1+(t^m)) \otimes k^\times \otimes R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ,$$

such that $\alpha - \varepsilon = \delta_r(\gamma)$ for some $\gamma \in (B_2(R_r) \otimes R_r^\times)^\circ$. We then define

$$\underline{\tilde{\Omega}}_{m,r}(\alpha) := \underline{L}_{m,r}(\gamma|_{t^m}) \in \Omega^1_{R/k}. \tag{6.2.2}$$

In order to see that $\underline{\tilde{\Omega}}_{m,r}(\alpha) := \underline{L}_{m,r}(\gamma|_{t^m}) \in \Omega^1_{R/k}$ is well-defined, suppose that

$$\varepsilon' \in im((1+(t^m)) \otimes k^\times \otimes R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$$

and $\gamma' \in (B_2(R_r) \otimes R_r^\times)^\circ$ are other such choices. Then

$$\delta_r(\gamma' - \gamma) = \varepsilon - \varepsilon' \in im((\Lambda^2 R_r^\times)^\circ \otimes k^\times) \subseteq \Lambda^3 R_r^\times$$

and hence $\underline{L}_{m,r}((\gamma' - \gamma)|_{t^m}) = 0$ by Lemma 6.2.5. The following proposition is the statement which we were looking for, that relates $\underline{\Omega}_{m,r}$ to $\underline{\tilde{\Omega}}_{m,r}$ and hence to $\underline{L}_{m,r}$:

Proposition 6.2.6. *Assuming Conjecture 5.2.5, we have* $\underline{\Omega}_{m,r} = \underline{\tilde{\Omega}}_{m,r}$.

Proof. Suppose that $x \ge m$ and $x+y+z = r$, with $y, z \ge 0$. We need to check that

$$\underline{\tilde{\Omega}}_{m,r}(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)) = a(yb \cdot dc - zc \cdot db).$$

(i) Case when $y = z = 0$. In this case, we need to compute the image of $\exp(at^r) \wedge \beta \wedge \gamma$ under $\underline{\tilde{\Omega}}_{m,r}$, where $\beta, \gamma \in R^\times$. The image of $\exp(at^r) \wedge \beta$ in $\oplus_{1\le i\le r-1} t^i\Omega^1_R$ is equal to 0. Therefore, there is $\alpha \in B_2^\circ(R_r)$ such that $\delta_r(\alpha) = \exp(at^r) \wedge \beta$. Then, by definition,

$$\underline{\tilde{\Omega}}_{m,r}(\exp(at^r) \wedge \beta \wedge \gamma) = \underline{L}_{m,r}((\alpha \otimes \gamma)|_{t^m}). \tag{6.2.3}$$

On the other hand, by (5.2.1), $L_{m,r}(\alpha|_{t^m} \otimes \gamma) = \ell i_{m,r}(\alpha|_{t^m})\frac{d\gamma}{\gamma}$. By the expression (3.0.2) for $\ell i_{m,r}$, we have

$$\ell i_{m,r}(\alpha|_{t_m}) = \Big(\Lambda^2 \log^\circ(\delta_r(\alpha))| \sum_{1\le i\le r-m} it^{r-i} \wedge t^i\Big) = 0,$$

since $\Lambda^2 \log^\circ(\delta_r(\alpha)) = \Lambda^2 \log^\circ(\exp(at^r) \wedge \beta) = 0$. By the above formula (6.2.3), this implies that $\underline{\tilde{\Omega}}_{m,r}(\exp(at^r) \wedge \beta \wedge \gamma) = 0$ as we wanted to show.

(ii) Case when $y \ne 0$ and $z = 0$. In this case we try to compute the image of $\exp(at^x) \wedge \exp(bt^y) \wedge \gamma$ under $\underline{\tilde{\Omega}}_{m,r}$. Here we assume that $\gamma \in R^\times$ and $x+y = r$, with $x \ge m$. By exactly the same argument as above, we deduce that there exists $\alpha \in B_2^\circ(R_r)$ such that $\delta_r(\alpha) = \exp(at^x) \wedge \exp(bt^y)$ and we have

$$\underline{\tilde{\Omega}}_{m,r}(\exp(at^x) \wedge \exp(bt^y) \wedge \gamma) = \underline{L}_{m,r}((\alpha \otimes \gamma)|_{t^m}) = \ell i_{m,r}(\alpha|_{t^m})\frac{d\gamma}{\gamma}.$$

Since $\delta_r(\alpha) = \exp(at^x) \wedge \exp(bt^y)$

$$\ell i_{m,r}(\alpha|_{t_m}) = \Big(\Lambda^2 \log^\circ(\delta_r(\alpha))| \sum_{1\le i\le r-m} it^{r-i} \wedge t^i\Big) = yab.$$

This exactly coincides with the expression in the statement of the proposition.

(iii) Case when $y \ne 0$ and $z \ne 0$. Note that, by localizing, we may assume that R is local. Moreover, since both sides of the expression are linear in a, b and c, we may assume without loss of generality that $a, b, c \in R^\times$. Since any element in a local ring can be written as a sum of units.

If $\theta := \exp(at^x) \wedge \exp(bt^y)$ then its image in $\oplus_{1\le i\le r-1} t^i\Omega^1_R$ is equal to

$$\oplus_{1\le i\le r-1} res_t \frac{1}{t^i}(\Lambda^2 d\log(\exp(at^x) \wedge \exp(bt^y))),$$

which only has a non-zero component in degree $x+y$ equal to $yb\cdot da - xa\cdot db$.

If we compute the image of $\varphi := \frac{x}{x+y}\exp(abt^{x+y})\wedge b - \frac{y}{x+y}\exp(abt^{x+y})\wedge a$ in the same group, we obtain the same element. Therefore $\theta-\varphi$ lies in the image of $B_2(R_r)$. Suppose that $\gamma_0\in B_2(R_r)$ such that $\delta(\gamma_0)=\theta-\varphi$. Since $\exp(abt^{x+y})\wedge\exp(ct^z)$ has weight r, there is $\varepsilon_0\in B_2(R_r)$ such that $\delta(\varepsilon_0)=\exp(abt^{x+y})\wedge\exp(ct^z)$. We now write

$$\begin{aligned}\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z) &= (\theta-\varphi)\wedge\exp(ct^z)+\varphi\wedge\exp(ct^z)\\ &= \delta(\gamma_0\otimes\exp(ct^z)) - \frac{x}{x+y}\delta(\varepsilon_0\otimes b)+\frac{y}{x+y}\delta(\varepsilon_0\otimes a).\end{aligned}$$

By the definition of $\underline{\tilde{\Omega}}_{m,r}$, we have $\underline{\tilde{\Omega}}_{m,r}(\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z))=$

$$\underline{L}_{m,r}(\gamma_0|_{t^m}\otimes\exp(ct^z)) - \frac{x}{x+y}\underline{L}_{m,r}(\varepsilon_0|_{t^m}\otimes b)+\frac{y}{x+y}\underline{L}_{m,r}(\varepsilon_0|_{t^m}\otimes a).$$

By definition, $\underline{L}_{m,r}(\varepsilon_0|_{t^m}\otimes b)=$

$$\ell i_{m,r}(\varepsilon_0|_{t^m})d\log(b) = (\Lambda^2\log^\circ\delta(\varepsilon_0)|\sum_{m\le i<r}(r-i)t^i\wedge t^{r-i})d\log(b) = zabc\cdot d\log(b) = azc\cdot db.$$

By the same argument, $\underline{L}_{m,r}(\varepsilon_0|_{t^m}\otimes a)=bzc\cdot da$.

In order to compute $\underline{L}_{m,r}(\gamma_0|_{t^m}\otimes\exp(ct^z))$, first note that, by the definition of $L_{m,r}$, we have

$$\underline{L}_{m,r}(\gamma_0|_{t^m}\otimes\exp(ct^z)) = -\frac{z}{x+y}d\ell i_{m,x+y}(\gamma_0|_{t^m})\cdot c + \ell i_{m,x+y}(\gamma_0|_{t^m})\cdot dc.$$

Since $\ell i_{m,x+y}(\gamma_0|_{t^m}) = (\Lambda^2\log^\circ\delta(\gamma_0)|\sum_{m\le i<x+y}(x+y-i)t^i\wedge t^{x+y-i}) = yab$, we have

$$\underline{L}_{m,r}(\gamma_0|_{t^m}\otimes\exp(ct^z)) = -\frac{zy}{x+y}c\cdot d(ab)+yab\cdot dc.$$

Combining all of these gives, $\underline{\tilde{\Omega}}_{m,r}(\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z))=$

$$-\frac{zy}{x+y}c\cdot d(ab)+yab\cdot dc-\frac{xzac}{x+y}\cdot db+\frac{yzbc}{x+y}da = a(yb\cdot dc - zc\cdot db).$$

This finishes the proof of the proposition. □

6.3. **Behaviour of $\omega_{m,r}$ with respect to automorphisms of R_{2m-1} which are identity modulo (t^m).** In this section, we continue to assume that R is smooth of relative dimension 1 over k. We will show the invariance of $\underline{\Omega}_{m,r}$ with respect to reparametrizations of R_r that are identity on the reduction to R_m. In order to do this, we will need to make an explicit computation on $k'((s))_\infty$, where k' is a finite extension on k. In order to make the formulas concise and intuitive, we will use several notational conventions as follows. If $a\in k'((s))$, we let $a' = a^{(1)}\in k'((s))$ denote its derivative with respect to s and $a^{(n+1)}=(a^{(n)})'$. Similarly, we let $\exp(a)$ denote an arbitrary *non-zero* element in $k'((s))$ and $a'=a^{(1)} := \frac{(\exp(a))'}{\exp(a)}$ and $a^{(n+1)}=(a^{(n)})'$. This notation is intuitive in the sense that, if one thinks of a as $\log(\exp(a))$ then a' is the logarithmic derivative of $\exp(a)$. With these conventions, we will state the following basic lemma.

Lemma 6.3.1. *Let σ be the automorphism of the k_∞ algebra $k'((s))_\infty$, which is the identity automorphism modulo (t) and has the property that $\sigma(s)=s+\alpha t^w$, with $w\ge 1$ and $\alpha\in k((s))$, then $\sigma(\exp(at^x)) = \exp(\sum_{0\le i}\frac{\alpha^i a^{(i)}}{i!}t^{x+iw})$.*

Proof. Assume that σ is such an automorphism of $k'((s))_\infty$. Let $i: k'\to k'((s))_\infty$ be the standard inclusion, which sends an element of k' to a constant series in s and t. In other words, i is the k'-algebra structure map. Let

$$\pi: k'((s))_\infty \to k'((s))_\infty/(t) = k'((s))$$

denote the canonical projection. By the assumptions, the restriction of $\sigma\circ i$ to k, is the standard inclusion $k\to k'((s))_\infty$. Similarly, by the assumptions, $\pi\circ\sigma\circ i$ is the standard inclusion of k' into $k'((s))$. Since k' is étale over k, these two statements above imply that $\sigma\circ i$ is the same as the inclusion i. Together with the assumption that $\sigma(t)=t$, this implies that σ is an automorphism of k'_∞-algebras.

The rest proof is then separated into two cases, when $x = 0$ and when $x \neq 0$. In both cases, the statement follows from the Taylor expansion formula. □

Lemma 6.3.2. *Let σ be the automorphism of the k_∞ algebra $k'((s))_\infty$, which is the identity automorphism modulo (t) and has the property that $\sigma(s) = s + \alpha t^w$, with $m \leq w$. Then we have,*

$$\underline{\Omega}_{m,r}\left(\frac{\sigma(\exp(at^i) \wedge \exp(bt^j) \wedge \exp(ct^k))}{\exp(at^i) \wedge \exp(bt^j) \wedge \exp(ct^k)}\right) = 0,$$

for $0 < i + j + k$.

Proof. Since $m \leq w$, $0 < i + j + k$ and $m < r < 2m$, the weight r terms of

$$\frac{\sigma(\exp(at^i) \wedge \exp(bt^j) \wedge \exp(ct^k))}{\exp(at^i) \wedge \exp(bt^j) \wedge \exp(ct^k)}$$

are possibly non-zero only when $i + j + k + w = r$ and in this case they are given by

$$\exp(\alpha a' t^{i+w}) \wedge \exp(bt^j) \wedge \exp(ct^k) + \exp(at^i) \wedge \exp(\alpha b' t^{j+w}) \wedge \exp(ct^k) \\ + \exp(at^i) \wedge \exp(bt^j) \wedge \exp(\alpha c' t^{k+w}).$$

By the definition of $\underline{\Omega}_{m,r}$, the above sum is sent to

$$\alpha(a'(jbc' - kcb') - b'(iac' - kca') + c'(iab' - jba'))ds = 0.$$

□

Corollary 6.3.3. *Let σ be any automorphism of R_r as a k_r-algebra, which reduces to identity on R_m, then $\omega_{m,r} \circ \Lambda^3\sigma = \omega_{m,r}$.*

Proof. This follows by the corresponding statement for $\underline{\Omega}_{m,r}$. This in turn reduces to Lemma 6.3.2 after localizing and completing. □

Definition 6.3.4. *If $\mathcal{R}/k_r$ is a smooth k_r-algebra of relative dimension 1. We defined the map*

$$\omega_{m,r} : \Lambda^3(\underline{\mathcal{R}}_r, (t^m))^\times \to \Omega^1_{\underline{\mathcal{R}}/k},$$

as the composition $\Omega_{m,r} \circ s$, where $\underline{\mathcal{R}}$ is the reduction of $\mathcal{R}$ modulo (t) and $\underline{\mathcal{R}}_r := \underline{\mathcal{R}} \times_k k_r$. Let $\tau : \underline{\mathcal{R}}_r \to \mathcal{R}$ be a splitting, that is an isomorphism of k_r-algebras which is the identity map modulo (t). By transport of structure, this gives a map

$$\omega_{m,r,\tau} : \Lambda^3(\mathcal{R}, (t^m))^\times \to \Omega^1_{\underline{\mathcal{R}}/k}.$$

Suppose that τ' is another such splitting which agrees with τ modulo (t^m). Applying Corollary 6.3.3 to ${\tau'}^{-1} \circ \tau$, we deduce that $\omega_{m,r,\tau} = \omega_{m,r,\tau'}$. Therefore, if $\sigma : \underline{\mathcal{R}}_m \to \mathcal{R}/(t^m)$ is a splitting of the reduction $\mathcal{R}/(t^m)$ of $\mathcal{R}$ then $\omega_{m,r,\sigma}$ is unambiguously defined as $\omega_{m,r,\tau}$, where τ is any splitting of $\mathcal{R}$ that reduces to σ modulo (t^m).

Recall the relative version of the Bloch group from [21, §2.4.8]. If A is a ring with ideal I, let $(A, I)^\flat := \{(\tilde{a}, \hat{a}) \in (A, I)^\times | (1 - \tilde{a}, 1 - \hat{a}) \in (A, I)^\times\}$. Then the relative Bloch group $B_2(A, I)$ is defined as the abelian group generated by the symbols $[(\tilde{a}, \hat{a})]$ for every $(\tilde{a}, \hat{a}) \in (A, I)^\flat$, modulo the relations generated by the analog of the five term relation for the dilogarithm:

$$[(\tilde{x}, \hat{x})] - [(\tilde{y}, \hat{y})] + [(\tilde{y}/\tilde{x}, \hat{y}/\hat{x})] - [(\frac{1 - \tilde{x}^{-1}}{1 - \tilde{y}^{-1}}, \frac{1 - \hat{x}^{-1}}{1 - \hat{y}^{-1}})] + [(\frac{1 - \tilde{x}}{1 - \tilde{y}}, \frac{1 - \hat{x}}{1 - \hat{y}})]$$

for every $(\tilde{x}, \hat{x}), (\tilde{y}, \hat{y}) \in (A, I)^\flat$ such that $(\tilde{x} - \tilde{y}, \hat{x} - \hat{y}) \in (A, I)^\times$. As in the classical case, we obtain a complex $\delta : B_2(A, I) \to \Lambda^2(A, I)^\times$, which sends $(\tilde{a}, \hat{a})$ to $(1 - \tilde{a}, 1 - \hat{a}) \wedge (\tilde{a}, \hat{a})$. As usual, abusing the notation, we will denote the induced map $\delta \otimes id : B_2(A, I) \otimes (A, I)^\times \to \Lambda^3(A, I)^\times$, also by δ. With these definitions, we have the following expected property of the map $\omega_{m,r,\tau}$.

Proposition 6.3.5. *For a splitting σ of $\mathcal{R}/(t^m)$, the above map $\omega_{m,r,\sigma}$ vanishes on the image of $B_2(\mathcal{R}, (t^m)) \otimes (\mathcal{R}, (t^m))^\times$ in $\Lambda^3(\mathcal{R}, (t^m))^\times$ under δ.*

Proof. By the definition of $\omega_{m,r,\sigma}$, we easily reduce to the split case where $\mathcal{R} = R_r$. It suffices to prove that $\Omega_{m,r}$ vanishes on the following two types of elements:

$$\delta_r([\hat{f}] \otimes \hat{g}) - \delta_r([\tilde{f}] \otimes \hat{g}) \quad \text{and} \quad \delta_r([\hat{f}] \otimes \hat{g}) - \delta_r([\hat{f}] \otimes \tilde{g}),$$

where $\hat{f},\, \tilde{f} \in \mathcal{R}^\flat$ have the same reduction modulo (t^m) and $\hat{g},\, \tilde{g} \in \mathcal{R}^\times$ have the same reduction modulo (t^m). This precisely the statement of Proposition 6.1.3. □

6.4. **Behaviour of** $res(\omega_{m,r,\sigma})$ **with respect to automorphisms of** R_m**.** In order to proceed with our construction, we need an object such as the 1-form in [21] which controls the effect of changing splittings. This object in $\star$-weight r will be constructed below by using $\omega_{m,r}$. On the other hand, this objects *does depend* on the choice of splittings if these splittings are different modulo (t^m), when $r > m+1$. In the modulus $m = 2$ case the only possible r is 3 so this situation does not occur in [21]. In the current case of higher modulus, we will see that the *residues* of the 1-form $\omega_{m,r}$ is invariant under the automorphisms of R_m which are identity modulo (t), which will imply that the residue can be defined independent of various choices. We will see that this will be enough for constructing the Chow dilogarithm of higher modulus. We will again start with an explicit computation on $k'((s))_\infty$.

Proposition 6.4.1. *Suppose that σ is the automorphism of $k'((s))_\infty$ as a k_∞-algebra such that $\sigma(s) = s + \alpha t^w$, with $w \geq 1$ and $\alpha \in k'((s))$, and which is identity modulo (t). Consider the element* $\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)$, *with $m \leq x$. If $r - (x + y + z) > 0$, and is divisible by w, let $q = \frac{r-(x+y+z)}{w}$. Then $\underline{\Omega}_{m,r}\big(\frac{\sigma(\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z))}{\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z)}\big)$ is equal to*

$$d\Big(\frac{\alpha^q}{q!} \sum_{0\leq k\leq q-1} a^{(k)} \binom{q-1}{k} \sum_{i+j=q-k} \Big(\binom{q-k-1}{i} y b^{(i)} c^{(j)} - \binom{q-k-1}{j} z b^{(i)} c^{(j)}\Big)\Big). \tag{6.4.1}$$

Otherwise, $\underline{\Omega}_{m,r}\big(\frac{\sigma(\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z))}{\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z)}\big) = 0.$

Proof. First note that by Lemma 6.3.1

$$\begin{aligned} Q &:= \frac{\sigma(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z))}{\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)} \\ &= \frac{\exp(\sum_{0\leq i} \frac{\alpha^i a^{(i)}}{i!} t^{x+iw}) \wedge \exp(\sum_{0\leq i} \frac{\alpha^i b^{(i)}}{i!} t^{y+iw}) \wedge \exp(\sum_{0\leq i} \frac{\alpha^i c^{(i)}}{i!} t^{z+iw})}{\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)} \end{aligned}$$

and hence if $r - (x + y + z) \leq 0$ or $w \nmid r - (x + y + z)$ then Q does not have a component of weight r and $\underline{\Omega}_{m,r}(Q) = 0$.

Suppose then that $r - (x + y + z) > 0$, $w|(r - (x + y + z))$ and let $q := \frac{r-(x+y+z)}{w}$ as in the statement of the proposition. In this case the weight r term of Q is given as

$$\sum_{i+j+k=q} \exp(\frac{\alpha^k a^{(k)}}{k!} t^{x+kw}) \wedge \exp(\frac{\alpha^i b^{(i)}}{i!} t^{y+iw}) \wedge \exp(\frac{\alpha^j c^{(j)}}{j!} t^{z+jw}).$$

This implies that $\underline{\Omega}_{m,r}(Q)$ is equal to

$$\sum_{i+j+k=q} \frac{\alpha^k a^{(k)}}{k!} \Big((y + iw) \frac{\alpha^i b^{(i)}}{i!} \big(\frac{\alpha^j c^{(j)}}{j!}\big)' - (z + jw) \big(\frac{\alpha^i b^{(i)}}{i!}\big)' \frac{\alpha^j c^{(j)}}{j!}\Big) ds.$$

We first claim that the expression above does not depend on w. The coefficient of $w \cdot \frac{a^{(k)}}{k!} \alpha^{q-1} d\alpha$ in this expression is $\sum_{i+j=q-k} (i \frac{b^{(i)}}{i!} \frac{jc^{(j)}}{j!} - j \frac{ib^{(i)}}{i!} \frac{c^{(j)}}{j!}) = 0$. The coefficient of $w \cdot \frac{a^{(k)}}{k!} \alpha^q ds$ in the same expression is

$$\sum_{i+j=q-k} (i \frac{b^{(i)}}{i!} \frac{c^{(j+1)}}{j!} - j \frac{b^{(i+1)}}{i!} \frac{c^{(j)}}{j!}) = \sum_{\substack{i+j=q-k+1 \\ 1\leq i,\, j}} \frac{b^{(i)}}{(i-1)!} \frac{c^{(j)}}{(j-1)!} - \sum_{\substack{i+j=q-k+1 \\ 1\leq i,\, j}} \frac{b^{(i)}}{(i-1)!} \frac{c^{(j)}}{(j-1)!} = 0.$$

Therefore $\underline{\Omega}_{m,r}(Q)$ can be rewritten as

$$\sum_{i+j+k=q} \frac{\alpha^k a^{(k)}}{k!}\Big(y\frac{\alpha^i b^{(i)}}{i!}\big(\frac{\alpha^j c^{(j)}}{j!}\big)' - z\big(\frac{\alpha^i b^{(i)}}{i!}\big)'\frac{\alpha^j c^{(j)}}{j!}\Big)ds. \tag{6.4.2}$$

The coefficient of $\alpha'\alpha^{q-1}ds$ in the above expression is equal to

$$\sum_{0\leq k\leq q-1}\frac{a^{(k)}}{k!}\sum_{i+j=q-k}\big(y\frac{b^{(i)}}{i!}\frac{jc^{(j)}}{j!} - z\frac{ib^{(i)}}{i!}\frac{c^{(j)}}{j!}\big)$$

which agrees with the coefficient of $\alpha'\alpha^{q-1}ds$ in (6.4.1).

Fix i_0, j_0, and k_0 such that $i_0+j_0+k_0=q$. Then the coefficient of $y\alpha^q a^{(k_0)}b^{(i_0)}c^{(j_0+1)}$ in (6.4.1) is equal to $\frac{1}{q}(\frac{1}{(k_0-1)!}\frac{1}{i_0!}\frac{1}{j_0!}+\frac{1}{k_0!}\frac{1}{i_0!}\frac{1}{(j_0-1)!}+\frac{1}{k_0!}\frac{1}{(i_0-1)!}\frac{1}{j_0!}) = \frac{1}{k_0!}\frac{1}{i_0!}\frac{1}{j_0!}$, which is exactly the same as the coefficient of the same term in (6.4.2). By symmetry, we deduce the same statement for the coefficients of $z\alpha^q a^{(k_0)}b^{(i_0+1)}c^{(j_0)}$. This finishes the proof of the proposition. □

Corollary 6.4.2. *Suppose that σ and $\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z)$ are as above. If $r=m+1$, then $\underline{\Omega}_{m,r}(\frac{\sigma(\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z))}{\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z)})=0$.*

Proof. In this case in order to have $m\leq x$ and $(m+1)-(x+y+z)=r-(x+y+z)>0$, we have to have $x=m$ and $y=z=0$. In this case, (6.4.1) is equal to 0. □

Corollary 6.4.3. *If $\mathcal{R}$ is a smooth k_{m+1}-algebra of relative dimension 1 as above, then for $r=m+1$, we have a well-defined map*

$$\omega_{m,m+1}:\Lambda^3(\mathcal{R},(t^m))^\times\to\Omega^1_{\underline{\mathcal{R}}/k}$$

as in Definition 6.3.4, which does not depend on the choice of a splitting of $\mathcal{R}/(t^m)$.

Proof. This follows immediately from Corollary 6.4.2, by reducing to the case $\mathcal{R}=k'((s))_{m+1}$, after localising and completing. □

For a general r between m and $2m$, the following corollary will be essential.

Corollary 6.4.4. *Fix $m<r<2m$, and let $\mathcal{R}$ be a smooth k_r-algebra of relative dimension 1 as above. Let x be a closed point of the spectrum of $\mathcal{R}$, k' its residue field, and let η be the generic point of $\mathcal{R}$. Then for any two splittings σ and σ' of $\mathcal{R}_\eta/(t^m)$, the reduction modulo (t^m) of the local ring of $\mathcal{R}$ at η, and for any $\alpha\in\Lambda^3(\mathcal{R}_\eta,(t^m))^\times$, the residues of $\omega_{m,r,\sigma}(\alpha)$ and $\omega_{m,r,\sigma'}(\alpha)\in\Omega^1_{\underline{\mathcal{R}}_\eta/k}$ at x are the same:*

$$res_x\omega_{m,r,\sigma'}(\alpha)=res_x\omega_{m,r,\sigma}(\alpha)\in k'.$$

Proof. Again by localising and completing we reduce to the case of $k'((s))_r$. By Proposition 6.4.1, we see that the difference $\omega_{m,r,\sigma'}(\alpha)-\omega_{m,r,\sigma}(\alpha)$ is the differential of an element in $k'((s))$ and hence has zero residue. □

Remark 6.4.5. Let $\mathcal{R}/k_r$ be as above. Suppose that τ and σ are two splittings $\underline{\mathcal{R}}_m\to\mathcal{R}/(t^m)$. In this case, there should be a map

$$h\omega_{m,r}(\tau,\sigma):\Lambda^3(\mathcal{R},(t^m))^\times\to\underline{\mathcal{R}}$$

such that

$$d(h\omega_{m,r}(\tau,\sigma))=\omega_{m,r,\tau}-\omega_{m,r,\sigma}.$$

Moreover, $h\omega_{m,r}(\tau,\sigma)$ should vanish on the image of $B_2(\mathcal{R},(t^m))\otimes(\mathcal{R},(t^m))^\times$.

In case $r=m+1$, $h\omega_{m,m+1}=0$ does satisfy the properties above. Let us look at the first non-trivial case when $m=3$ and $r=5$. Note that the reduction modulo (t^2) of the automorphism $\tau^{-1}\circ\sigma:\underline{\mathcal{R}}_3\to\underline{\mathcal{R}}_3$, which lifts the identity map on $\underline{\mathcal{R}}$, is determined by a k-derivation $\theta:\underline{\mathcal{R}}\to\underline{\mathcal{R}}$.

Define $h\underline{\Omega}_{3,5}(\theta):I_{3,5}\subseteq(\Lambda^3\underline{\mathcal{R}}_5^\times)^\circ\to\underline{\mathcal{R}}$, as

$$h\underline{\Omega}_{3,5}(\theta)(\exp(at^3)\wedge\exp(bt)\wedge c)=ab\theta(\frac{dc}{c}),$$

where a, b, $\in \underline{\mathcal{R}}$ and $c \in \underline{\mathcal{R}}^{\times}$. Let $h\underline{\Omega}_{3,5}(\theta)$ be defined as 0 on all the other type of elements in $I_{3.5}$. Then $h\omega_{3,5}(\tau,\sigma) : \Lambda^3(\mathcal{R},(t^3))^{\times} \to \underline{\mathcal{R}}$ defined by

$$h\omega_{3,5}(\tau,\sigma)(\alpha) := -h\underline{\Omega}_{3,5}(\theta)(s(\sigma^{-1}(\alpha)))$$

satisfies the desired properties above. An analog of this construction is one of the main tools in defining an infinitesimal version of the Bloch regulator in [20].

Definition 6.4.6. *Let $\mathcal{R}$ be a smooth k_r-algebra of relative dimension 1 as above. Let η be the generic point and x be a closed point of the spectrum of $\mathcal{R}$. Then we have a canonical map*

$$res_x\omega_{m,r} : \Lambda^3(\mathcal{R}_\eta,(t^m))^{\times} \to k',$$

where k' is the residue field of x. The map is defined by choosing any splitting σ of $\mathcal{R}_\eta/(t^m)$ and letting $res_x\omega_{m,r} := res_x\omega_{m,r,\sigma}$. This is independent of the choice of the splitting σ, by Corollary 6.4.4.

6.5. **Variant of the residue map for different liftings.** For the construction of the infinitesimal Chow dilogarithm, we need a variant of Definition 6.4.6. Fortunately, we do not need to do extra work, Corollary 6.3.3 and Proposition 6.4.1 will still be sufficient to give us what we are looking for.

Suppose that A is a ring with an ideal I and B and B' are two A-algebras together with an isomorphism $\chi : B/IB \simeq B'/IB'$ of A-algebras. We let

$$(B,B',\chi)^{\times} := \{(p,p')|p \in B^{\times} \text{ and } p' \in B'^{\times} \text{ s.t. } \chi(p|_I) = p'|_I\},$$

where $p|_I$ denotes the image of p in $(B/IB)^{\times}$. Similarly, we define $(B,B',\chi)^{\flat}$ and $B_2(B,B',\chi)$ and obtain maps, $B_2(B,B',\chi) \to \Lambda^2(B,B',\chi)^{\times}$ and $B_2(B,B',\chi)\otimes(B,B',\chi)^{\times} \to \Lambda^3(B,B',\chi)^{\times}$. We will use these definitions below with $A = k_\infty$ and $I = (t^m)$. In fact the following variant will be essential in what follows.

Suppose that $\mathcal{S}/k_m$ is a smooth algebra of relative dimension 1, with x a closed point and η the generic point of its spectrum. Suppose that $\mathcal{R}$, $\mathcal{R}'/k_r$ are liftings of $\mathcal{S}_\eta$ to k_r. In other words, we have fixed isomorphisms:

$$\psi : \mathcal{R}/(t^m) \to \mathcal{S}_\eta$$

and

$$\psi' : \mathcal{R}'/(t^m) \to \mathcal{S}_\eta.$$

Letting $\chi := \psi'^{-1} \circ \psi$, we would like to construct a map

$$res_x\omega_{m,r} : \Lambda^3(\mathcal{R},\mathcal{R}',\chi)^{\times} \to k',$$

where k' is the residue field of x. Note that $(\mathcal{R},\mathcal{R}',\chi)^{\times}$ consists of pairs of (p,p') with $p \in \mathcal{R}^{\times}$ and $p' \in \mathcal{R}'^{\times}$ such that $\psi(p|_{t^m}) = \psi'(p'|_{t^m})$. In other words, it consists of different liftings of elements of $\mathcal{S}_\eta^{\times}$. We sometimes use the notation $(\mathcal{R},\mathcal{R}',\psi,\psi')^{\times}$ to denote the same set.

In order to construct this map, let

$$\tilde{\chi} : \mathcal{R} \to \mathcal{R}'$$

be an isomorphism of k_r-algebras which is a lifting of χ. This provides us with a map

$$(\mathcal{R},\mathcal{R}',\chi)^{\times} \xrightarrow{\tilde{\chi}^*} (\mathcal{R},(t^m))^{\times}.$$

Choosing a splitting $\sigma : \underline{\mathcal{R}}_m \to \mathcal{R}/(t^m)$, by Definition 6.3.4 we obtain the map $\omega_{m,r,\sigma}$, composing this with the map induced by the reduction $\underline{\psi}$ of ψ, we obtain

$$\Lambda^3(\mathcal{R},\mathcal{R}',\chi)^{\times} \xrightarrow{\Lambda^3\tilde{\chi}^*} \Lambda^3(\mathcal{R},(t^m))^{\times} \xrightarrow{\omega_{m,r,\sigma}} \Omega^1_{\underline{\mathcal{R}}/k} \xrightarrow{d\underline{\psi}} \Omega^1_{\underline{\mathcal{S}}_\eta/k} \xrightarrow{res_x} k'. \tag{6.5.1}$$

Proposition 6.5.1. *The map (6.5.1) above is independent of the choices of the lifting $\tilde{\chi}$ of χ and the choice of the splitting σ of $\mathcal{R}/(t^m)$.*

Proof. That the composition is independent of the choice of $\tilde{\chi}$ follows from Corollary 6.3.3 and Definition 6.3.4. That it is independent of the choice of the splitting σ follows from Proposition 6.4.1. □

Definition 6.5.2. *We denote the composition (6.5.1) above by*

$$res_x\omega_{m,r}(\psi,\psi') : \Lambda^3(\mathcal{R},\mathcal{R}',\psi,\psi')^\times \to k'.$$

If ψ and ψ' are clear from the context, we denote this map by $res_x\omega_{m,r}$, and $(\mathcal{R},\mathcal{R}',\psi,\psi')^\times$ by $(\mathcal{R},\mathcal{R}',(t^m))^\times$. Depending on the context, we also use the notation $res_x\omega_{m,r}(\chi) : \Lambda^3(\mathcal{R},\mathcal{R}',\chi) \to k'$ for the same map, with $\chi = \psi'^{-1} \circ \psi$.

With these definitions, we have the following corollary.

Corollary 6.5.3. *Suppose that $\mathcal{R}$ and $\mathcal{R}'$ are smooth k_r-algebras of dimension 1 as above which are liftings of the generic local ring $\mathcal{S}_\eta$ of a smooth k_m-algebra $\mathcal{S}$. Let $\chi : \mathcal{R}/(t^m) \to \mathcal{R}'/(t^m)$ be the corresponding isomorphism of k_m-algebras. Let x be a closed point of $\underline{\mathcal{S}}$. Then the map*

$$res_x\omega_{m,r}(\chi) : \Lambda^3(\mathcal{R},\mathcal{R}',\chi)^\times \to k'$$

vanishes on the image of $B_2(\mathcal{R},\mathcal{R}',\chi) \otimes (\mathcal{R},\mathcal{R}',\chi)^\times$.

Proof. Follows from Proposition 6.3.5. □

7. The residue of $\omega_{m,r}$ on good liftings.

Suppose that $\mathcal{R}/k_r$ is as above. Moreover, we assume that the reduction $\underline{\mathcal{R}}$ of $\mathcal{R}$ modulo (t) is a discrete valuation ring with x being the closed point. We let $\tilde{x}/k_r$ be a *lifting* of x to $\mathcal{R}$. By this what we mean is as follows. Let s be a uniformizer at x, and let $\tilde{s}$ be any lifting of s to $\mathcal{R}$, we call $\tilde{s}$ also a uniformizer at x on $\mathcal{R}$. The associated scheme $\tilde{x}$, which is smooth over k_r, is what we call a lifting of x. In other words a lifting of x is a 0-dimensional closed subscheme $\tilde{x}$ of $\mathcal{R}$ such that its ideal is generated by a single element which reduces to a uniformizer on $\underline{\mathcal{R}}$. Note that if we are given $\tilde{x}$, then $\tilde{s}$ is determined up to a unit in $\mathcal{R}$. Sometimes we will abuse the notation and write $(\tilde{s})$ instead of $\tilde{x}$. Let η denote the generic point of $\mathcal{R}$. We let

$$(\mathcal{R},\tilde{x})^\times := \{\alpha \in \mathcal{R}_\eta^\times | \alpha = u\tilde{s}^n, \text{ for some } u \in \mathcal{R}^\times \text{ and } n \in \mathbb{Z}\}.$$

We say that an element $\alpha \in \mathcal{R}_\eta^\times$ is *good with respect to* $\tilde{x}$, if $\alpha \in (\mathcal{R},\tilde{x})^\times$. Note that this property depends only on $\tilde{x}$, and not on $\tilde{s}$. The importance of this notion for us is that for wedge products of good liftings, we can define their residue along (s) as in [21, §2.4.5]. Namely, there is a map

$$res_{\tilde{x}} : \Lambda^n(\mathcal{R},\tilde{x})^\times \to \Lambda^{n-1}(\mathcal{R}/(s))^\times,$$

with the properties that it vanishes on $\Lambda^n\mathcal{R}^\times$ and $s\wedge\alpha_1\wedge\cdots\wedge\alpha_{n-1}$ is mapped to $\underline{\alpha}_1\wedge\cdots\wedge\underline{\alpha}_{n-1}$, if $\alpha_i \in \mathcal{R}^\times$ and $\underline{\alpha}_i$ denotes the image of α_i in $(R/(s))^\times$, for $1 \le i \le n-1$.

Suppose that $\mathcal{R}'/k_r$ is another such ring, and $\tilde{x}'$ a lifting of the closed point of $\underline{\mathcal{R}}'$. Suppose that there is an isomorphism $\chi : \mathcal{R}/(t^m) \to \mathcal{R}'/(t^m)$ which identifies the reduction of $\tilde{x}$ modulo (t^m) with the reduction of $\tilde{x}'$ modulo (t^m). Then we let

$$(\mathcal{R},\mathcal{R}',\tilde{x},\tilde{x}',\chi)^\times := \{(p,p')|p \in (\mathcal{R},\tilde{x})^\times \text{ and } p' \in (\mathcal{R}',\tilde{x}')^\times \text{ such that } \chi(p|_{t^m}) = p'|_{t^m}\}.$$

Note that clearly $(\mathcal{R},\mathcal{R}',\tilde{x},\tilde{x}',\chi)^\times \subseteq (\mathcal{R},\mathcal{R}',\chi)^\times$. In case $\mathcal{R}' = \mathcal{R}$ with χ the identity map, we denote the corresponding group by $(\mathcal{R},\tilde{x},(t^m))^\times$. Denote the natural maps $(\mathcal{R},\mathcal{R}',\tilde{x},\tilde{x}',\chi)^\times \to (\mathcal{R},\tilde{x})^\times$ and $(\mathcal{R},\mathcal{R}',\tilde{x},\tilde{x}',\chi)^\times \to (\mathcal{R}',\tilde{x}')^\times$ by π_1 and π_2.

In this section, we would like to compute $res_x\omega_{m,r}(\chi)(\alpha)$ for $\alpha \in \Lambda^3(\mathcal{R},\mathcal{R}',\tilde{x},\tilde{x}',\chi)^\times$ in terms of the value of $\ell_{m,r}$ on the residue of α. The main result of this section is Proposition 7.0.3. We will first start with certain explicit computations on the formal power series rings and then finally reduce our general statement to these special cases. Let us immediately remark that in order to compute the residues, we immediately reduce to the case when $\mathcal{R}$ and $\mathcal{R}'$ are complete with respect to the ideal which correspond to their closed points.

We will first consider the case of $R = k'[[s]]$ and that of the same uniformizer on both of the liftings as follows.

Note that

$$res_{(s)}(s\wedge\alpha\wedge\beta) = \underline{\alpha}\wedge\underline{\beta} \in \Lambda^2 {k'_r}^\times,$$

where $\underline{\alpha}$ and $\underline{\beta}$ are the images of α and β under the natural projection $\mathcal{R}^\times \to (\mathcal{R}/(s))^\times = {k'_r}^\times$. Similarly for $\underline{p'}$.

Lemma 7.0.1. *Suppose that* $R = k'[[s]]$, *and* $\mathcal{R} := R_r = k'[[s,t]]/(t^r)$. *Suppose that* α, α', *and* $\beta \in \mathcal{R}^\times$ *such that* $\alpha'|_{t^m} = \alpha|_{t^m} \in \mathcal{R}/(t^m)$. *Let* $p' := s \wedge \alpha' \wedge \beta$, $p := s \wedge \alpha \wedge \beta$ *and* $(p,p') := (s,s) \wedge (\alpha, \alpha') \wedge (\beta, \beta) \in \Lambda^3(\mathcal{R}, (s), (t^m))^\times \subseteq \Lambda^3(\mathcal{R}_\eta, (t^m))^\times$. *Then the residue of* $\omega_{m,r}(p,p')$ *at the closed point of* $\underline{\mathcal{R}} = k'[[s]]$ *is given by*

$$res_{s=0}\omega_{m,r}(p,p') = \ell_{m,r}(res_{(s)}(p)) - \ell_{m,r}(res_{(s)}(p')).$$

Proof. By Definition 6.1.4, we see that $\omega_{m,r}(p,p') = \underline{\Omega}_{m,r}(p - p') = \underline{\Omega}_{m,r}(s \wedge \frac{\alpha}{\alpha'} \wedge \beta)$. Let us compute the residue at $s = 0$ of an expression of the type $\underline{\Omega}_{m,r}(s \wedge \exp(at^i) \wedge \exp(bt^j))$, with $a, b \in k[[s]]$ and $i \geq m$, such that if $j = 0$. We use the notation in Remark 6.1.2. Since

$$\underline{\Omega}_{m,r}(s \wedge \exp(at^i) \wedge \exp(bt^j)) = jab\frac{ds}{s},$$

when $i + j = r$ and is 0 otherwise, we conclude that its residue is equal to $ja(0)b(0)$ if $i + j = r$, and is 0 otherwise. Since, for $i \geq m$,

$$\ell_{m,r}(res_{(s)}(s \wedge \exp(at^i) \wedge \exp(bt^j))) = \ell_{m,r}(\exp(a(0)t^i) \wedge \exp(b(0)t^j))$$

is equal to $ja(0)b(0)$ if $i + j = r$, and is 0 otherwise, we conclude that

$$res_{s=0}(\underline{\Omega}_{m,r}(s \wedge \exp(at^i) \wedge \exp(bt^j))) = \ell_{m,r}(res_{(s)}(s \wedge \exp(at^i) \wedge \exp(bt^j))). \tag{7.0.1}$$

On the other hand, since $\alpha'|_{t^m} = \alpha|_{t^m}$, $s \wedge \frac{\alpha}{\alpha'} \wedge \beta$ is a sum of terms of the above type, and the linearity of both sides of (7.0.1) imply that (7.0.1) is also valid for $s \wedge \frac{\alpha}{\alpha'} \wedge \beta$. By linearity of $\ell_{m,r}$ and $res_{(s)}$, we have

$$\ell_{m,r}(res_{(s)}(s \wedge \frac{\alpha}{\alpha'} \wedge \beta)) = \ell_{m,r}(res_{(s)}(s \wedge \alpha \wedge \beta)) - \ell_{m,r}(res_{(s)}(s \wedge \alpha' \wedge \beta)),$$

which together with the above proves the lemma. □

Let us now try to prove the same formula when the choice of the uniformizer is not the same. In other words, with notation as above let $s' \in \mathcal{R}$ such that $s'|_{t^m} = s|_{t^m}$. For simplicity, let us temporarily use the notation $(\mathcal{R}, (s), (s'), (t^m))^\times := (\mathcal{R}, \mathcal{R}, (s), (s'), id_{\mathcal{R}/(t^m)})^\times$. Let $p' := s' \wedge \alpha \wedge \beta$, $p := s \wedge \alpha \wedge \beta$ and $(p,p') := (s,s') \wedge (\alpha, \alpha) \wedge (\beta, \beta) \in \Lambda^3(\mathcal{R}, (s), (s'), (t^m))^\times$.

Lemma 7.0.2. *With notation as above, the residue of* $\omega_{m,r}(p,p')$ *at the closed point of* $k'[[s]]$ *is given by the following formula:*

$$res_{s=0}\omega_{m,r}(p,p') = \ell_{m,r}(res_{(s)}(p)) - \ell_{m,r}(res_{(s')}(p')).$$

Proof. If s'' is another lift of the uniformizer s, in other words $s'' \in \mathcal{R}$ with $s''|_{t^m} = s|_{t^m}$ then

$$res_{s=0}\omega_{m,r}(p,p'') = res_{s=0}\omega_{m,r}(p,p') + res_{s=0}\omega_{m,r}(p',p'')$$

and $\ell_{m,r}(res_{(s)}(p)) - \ell_{m,r}(res_{(s'')}(p'')) =$

$$\big(\ell_{m,r}(res_{(s)}(p)) - \ell_{m,r}(res_{(s')}(p'))\big) + \big(\ell_{m,r}(res_{(s')}(p')) - \ell_{m,r}(res_{(s'')}(p''))\big).$$

Therefore in order to prove the lemma we may assume without loss of generality that $s' = s + at^i$, with $a \in k'[[s]]$ and $m \leq i$. Note that in $\mathcal{R}$, we have $s + at^i = s\exp(\frac{a}{s}t^i)$, since $r < 2m$. Letting $\alpha = \exp(bt^j)$ and $\beta = \exp(ct^k)$, we can rewrite $\omega_{m,r}(p,p')$ as

$$\underline{\Omega}_{m,r}(p - p') = \underline{\Omega}_{m,r}(\exp(-\frac{a}{s}t^i) \wedge \exp(bt^j) \wedge \exp(ct^k)) = -\frac{a}{s}(jb \cdot dc - kc \cdot db),$$

if $i + j + k = r$ and 0 otherwise. Its residue is

$$-a(0)(jb(0)c'(0) - kc(0)b'(0)) \tag{7.0.2}$$

if $i + j + k = r$ and 0 otherwise, with the usual conventions if j or k is 0.

On the other hand, $res_{(s)}(p) = \exp(b(0)t^j) \wedge \exp(c(0)t^k)$ and

$$res_{(s')}(p') = \exp(b(0)t^j - a(0)b'(0)t^{i+j}) \wedge \exp(c(0)t^k - a(0)c'(0)t^{i+k}) \in \Lambda^2 {k'_r}^\times.$$

By the linearity of $\ell_{m,r}$, the right hand side of the expression in the statement of the lemma is then equal to

$$-\ell_{m,r}(\exp(b(0)t^j) \wedge \exp(-a(0)c'(0)t^{i+k})) - \ell_{m,r}(\exp(-a(0)b'(0)t^{i+j}) \wedge \exp(c(0)t^k))$$

$$- \ell_{m,r}(\exp(-a(0)b'(0)t^{i+j}) \wedge \exp(-a(0)c'(0)t^{i+k})).$$

The last summand is equal to 0 since $\ell_{m,r}$ is of weight r and $i+j+i+k \geq 2i \geq 2m > r$. For the same reason, the first two summands are 0 if $i+j+k \neq r$ and if $i+j+k = r$, then the total expression is equal to $-a(0)c'(0)jb(0) + a(0)b'(0)kc(0)$, which agrees with the formula (7.0.2) for the residue of $\Omega_{m,r}$. Since α and β are sums of the terms of the above type, this proves the lemma. □

Proposition 7.0.3. *Suppose that $\mathcal{R}$, $\mathcal{R}'$ are local algebras which are smooth of relative dimension 1 over k_r, together with liftings $\tilde{x}$, $\tilde{x}'$ of their closed points and a k_m-isomorphism $\chi : \mathcal{R}/(t^m) \to \mathcal{R}'/(t^m)$ which maps the reductions of $\tilde{x}$ and $\tilde{x}'$ to each other. Then for $q \in \Lambda^3(\mathcal{R}, \mathcal{R}', \tilde{x}, \tilde{x}', \chi)^\times$, we have the following formula for the closed point x,*

$$res_x \omega_{m,r}(q) = \ell_{m,r}(res_{\tilde{x}}(\Lambda^3\pi_1)(q)) - \ell_{m,r}(res_{\tilde{x}'}(\Lambda^3\pi_2)(q)).$$

Proof. In order to prove the statement, we can replace $\mathcal{R}$ and $\mathcal{R}'$ with their completions at their closed points. Let k' be the residue field at the closed point x. Without loss of generality, we will assume that $\mathcal{R} = \mathcal{R}' = k'[[s]]_r$, $\tilde{x}$ is given by $s = 0$ and $\tilde{x}'$ is given by $s' = 0$ for some $s' \in k'[[s]]_r$ with $s'|_{t^m} = s|_{t^m}$, and χ is the map which is identity on $k'[[s]]_m$.

In order to make the computations we need to choose a lifting $\tilde{\chi}$ of χ from $\mathcal{R}$ to $\mathcal{R}'$. We choose this lifting to be the one that sends s to s' and is identity on k'. Note that $\tilde{\chi}$ being a map of k_r algebras has to satisfy $\tilde{\chi}(t) = t$.

The statement above then reduces to the following: suppose that $\alpha, \beta, \gamma \in (k'[[s]]_r, (s))^\times$ and $\alpha', \beta', \gamma' \in (k'[[s]]_r, (s'))^\times$ such that $\alpha|_{t^m} = \alpha'|_{t^m}$, $\beta|_{t^m} = \beta'|_{t^m}$, and $\gamma|_{t^m} = \gamma'|_{t^m}$, and $p = \alpha \wedge \beta \wedge \gamma$, $p' = \alpha' \wedge \beta' \wedge \gamma'$, and $(p, p') = (\alpha, \alpha') \wedge (\beta, \beta') \wedge (\gamma, \gamma')$, then

$$res_{s=0}\Omega_{m,r}(p - p') = \ell_{m,r}(res_{(s)}(p)) - \ell_{m,r}(res_{(s')}(p')).$$

By assumption α is of the form us^n for some $u \in k'[[s]]_r^\times$ and $n \in \mathbb{Z}$. Similarly, α' is of the form $u's'^{n'}$, with $u' \in k'[[s]]_r^\times$. The condition that $\alpha|_{t^m} = \alpha'|_{t^m}$ implies that $n = n'$. The same is true for β, β', and γ, γ'. By multi-linearity and anti-symmetry, we reduce to checking the above identity in the following two cases: in the first case where α, $\beta, \gamma \in k'[[s]]^\times$ and in the second case where $\alpha = s$, $\alpha' = s'$ and β, $\gamma \in k'[[s]]^\times$.

If α, $\beta, \gamma \in k'[[s]]^\times$, then α', $\beta', \gamma' \in k'[[s]]^\times$. This implies on the one hand that $res_{(s)}(p) = 0$ and $res_{(s')}(p') = 0$, and on the other that $p - p' \in I_{m,r} = (1 + (t^m) \otimes \Lambda^2 k'[[s]]_r^\times) \subseteq (\Lambda^3 k'[[s]]_r^\times)^\circ$, which implies that $\underline{\Omega}_{m,r}(p - p') \in \Omega^1_{k'[[s]]/k}$. Therefore $res_{s=0}\underline{\Omega}_{m,r}(p - p') = 0 = \ell_{m,r}(res_{(s)}(p)) - \ell_{m,r}(res_{(s')}(p'))$ in this case.

Let us now consider the more interesting case of $\alpha = s$, $\alpha' = s'$ and β, γ, β', $\gamma' \in k'[[s]]^\times$, with $\beta|_{t^m} = \beta'|_{t^m}$ and $\gamma|_{t^m} = \gamma'|_{t^m}$. Applying Lemma 7.0.1 first with $p = (s, \alpha, \beta)$ and $p' = (s, \alpha', \beta)$ then with $p = (s, \alpha', \beta)$ and $p' = (s, \alpha', \beta')$ and then applying Lemma 7.0.2 with $p = (s, \alpha', \beta')$ and $p' = (s', \alpha', \beta')$ and adding all the equalities finishes the proof of the proposition. □

8. Construction of ρ and a regulator on curves

8.1. **Regulators on curves.** Let $\mathcal{R}/k_m$ be smooth of relative dimension 1, as in the previous section but without the assumption that $\underline{\mathcal{R}}$ is a discrete valuation ring. Choose and fix a lifting $\mathfrak{c}$ of c to $\mathcal{R}$ for every closed point c of $\underline{\mathcal{R}}$ as in the previous section. We denote the set of these liftings by $\mathscr{P}$. We let $k(c)$ denote the residue field of c and $k(\mathfrak{c})$ denote the artin ring which is the ring of regular functions on the affine scheme $\mathfrak{c}$. Let $|\underline{\mathcal{R}}| = |\mathcal{R}|$ denote the set of closed points of $\underline{\mathcal{R}}$, or equivalently of $\mathcal{R}$. Note that the reductions of the localizations $\mathcal{R}_c$ of $\mathcal{R}$ are discrete valuation rings. We let

$$(\mathcal{R}, \mathscr{P})^\times := \bigcap_{c \in |\mathcal{R}|} (\mathcal{R}_c, \mathfrak{c})^\times$$

and $(\mathcal{R}, \mathscr{P})^\flat := \{f \in (\mathcal{R}, \mathscr{P})^\times | 1 - f \in (\mathcal{R}, \mathscr{P})^\times\}$. We define $B_2(\mathcal{R}, \mathscr{P})$ to be the vector space over $\mathbb{Q}$ generated by the symbols $[f]$ with $f \in (\mathcal{R}, \mathscr{P})^\flat$ modulo the five term relations associated to pairs f and g in $(\mathcal{R}, \mathscr{P})^\flat$ which have the property that $f - g \in (\mathcal{R}, \mathscr{P})^\times$. As usual we have maps $B_2(\mathcal{R}, \mathscr{P}) \to \Lambda^2(\mathcal{R}, \mathscr{P})^\times$ and $B_2(\mathcal{R}, \mathscr{P}) \otimes (\mathcal{R}, \mathscr{P})^\times \to \Lambda^3(\mathcal{R}, \mathscr{P})^\times$. We also have a residue map $res_{\mathfrak{c}}: B_2(\mathcal{R},\mathscr{P})\otimes(\mathcal{R},\mathscr{P})^\times \to B_2(k(\mathfrak{c}))$ that is defined exactly as in [21, §3.3.1] and which gives a commutative diagram:

$$\begin{array}{ccc} B_2(\mathcal{R},\mathscr{P})\otimes(\mathcal{R},\mathscr{P})^\times & \xrightarrow{\delta} & \Lambda^3(\mathcal{R},\mathscr{P})^\times \\ \downarrow{\scriptstyle res_{\mathfrak{c}}} & & \downarrow{\scriptstyle res_{\mathfrak{c}}} \\ B_2(k(\mathfrak{c})) & \xrightarrow{\delta} & \Lambda^2 k(\mathfrak{c})^\times. \end{array}$$

Suppose that C/k_m is a smooth and projective curve. For every closed point c of $\underline{C}$, choose and fix a smooth lifting of $\mathfrak{c}$ of c to C. We denote $\mathscr{P}$ to be the set of these liftings. We let $(\mathcal{O}_C,\underline{\mathscr{P}})^\times$ denote the sheaf on $\underline{C}$ which associates to an open set U of $\underline{C}$, the group $(\mathcal{O}_C(U),\mathscr{P}|_U)^\times$. Similarly, $B_2(\mathcal{O}_C,\underline{\mathscr{P}})$ is the sheaf associated to the presheaf, which associates to U the group $B_2(\mathcal{O}_C(U),\mathcal{P}|_U)$. For each $c\in|C|$, let i_c denote the imbedding of c in $\underline{C}$. The commutative diagram above gives us a complex $\underline{\Gamma}'_B(C,\mathscr{P},3)$ of sheaves:

$$B_2(\mathcal{O}_C,\underline{\mathscr{P}})\otimes(\mathcal{O}_C,\underline{\mathscr{P}})^\times \to \oplus_{c\in|C|} i_{c*}(B_2(k(\mathfrak{c})))\oplus\Lambda^3(\mathcal{O}_C,\underline{\mathscr{P}})^\times \to \oplus_{c\in|C|} i_{c*}(\Lambda^2 k(\mathfrak{c})^\times),$$

concentrated in degrees $[2,4]$. We use the following sign conventions in the above complex: the first map is (δ, res) and the second one is $-\delta+res$. We will be interested in the infinitesimal part of the degree 3 cohomology $\mathrm{H}^3_B(C,\mathbb{Q}(3)) := \mathrm{H}^3(\underline{C},\underline{\Gamma}'_B(C,\mathscr{P},3))$ of the complex $\underline{\Gamma}'_B(C,\mathscr{P},3)$. More precisely, we will be interested in defining regulator maps from $\mathrm{H}^3_B(C,\mathbb{Q}(3))$ to k for every $m<r<2m$.

The above cohomology group is a candidate for the motivic cohomology group $\mathrm{H}^3_{\mathcal{M}}(C,\mathbb{Q}(3))$. To be more precise, we would expect a sheaf $B_3(\mathcal{O}_C,\underline{\mathscr{P}})$ of Bloch groups of weight 3 as in [6], which would fit into a complex $\underline{\Gamma}_B(C,\mathscr{P},3)$ of sheaves on $\underline{C}$:

$$B_3(\mathcal{O}_C,\underline{\mathscr{P}})\to B_2(\mathcal{O}_C,\underline{\mathscr{P}})\otimes(\mathcal{O}_C,\underline{\mathscr{P}})^\times \to \oplus i_{c*}(B_2(k(\mathfrak{c})))\oplus\Lambda^3(\mathcal{O}_C,\underline{\mathscr{P}})^\times \to \oplus i_{c*}(\Lambda^2 k(\mathfrak{c})^\times),$$

and which would compute motivic cohomology of weight 3. Since we are only interested in $\mathrm{H}^3(\underline{C},\underline{\Gamma}_B(C,\mathscr{P},3))$ and since on a curve, by Grothendieck's vanishing theorem, the cohomology of any sheaf vanishes in degree greater than 1, we have an isomorphism

$$\mathrm{H}^3(\underline{C},\underline{\Gamma}'_B(C,\mathscr{P},3))\simeq\mathrm{H}^3(\underline{C},\underline{\Gamma}_B(C,\mathscr{P},3)).$$

For a sheaf of complexes $\mathscr{F}_\cdot$, let $\check{\mathrm{H}}^\cdot(\underline{C},\mathscr{F}_\cdot)$ denote the colimit of all the Čech cohomology groups over all Zariski covers of $\underline{C}$. For a sheaf $\mathscr{F}$, the natural map $\check{\mathrm{H}}^i(\underline{C},\mathscr{F})\to\mathrm{H}^i(\underline{C},\mathscr{F})$ is an isomorphism for $i=0$, 1. By the same argument, it follows that the same is true for a complex of sheaves $\mathscr{F}_\cdot$ which is concentrated in degrees 0 and 1. This applied to the complex above implies that the natural map

$$\check{\mathrm{H}}^3(\underline{C},\underline{\Gamma}'_B(C,\mathscr{P},3))\simeq\mathrm{H}^3(\underline{C},\underline{\Gamma}'_B(C,\mathscr{P},3))$$

is an isomorphism. Therefore, it is enough to construct the map $\check{\mathrm{H}}^3(\underline{C},\underline{\Gamma}'_B(C,\mathscr{P},3))\to k$. We will in fact construct the map as the composition

$$\check{\mathrm{H}}^3(\underline{C},\underline{\Gamma}'_B(C,\mathscr{P},3))\hookrightarrow\check{\mathrm{H}}^3(\underline{C},\underline{\Gamma}''_B(C,\mathscr{P},3))\to k,$$

where $\underline{\Gamma}''_B(C,\mathscr{P},3)$ is the quotient complex:

$$B_2(\mathcal{O}_C,\underline{\mathscr{P}})\otimes(\mathcal{O}_C,\underline{\mathscr{P}})^\times \to \oplus_{c\in|C|} i_{c*}(B_2(k(\mathfrak{c})))\oplus\Lambda^3(\mathcal{O}_C,\underline{\mathscr{P}})^\times$$

of $\underline{\Gamma}'_B(C,\mathscr{P},3)$.

Suppose that we are given a Zariski open cover $U_\cdot$ of $\underline{C}$, we will define a map from the corresponding cocycle group $\check{\mathrm{Z}}^3(U_\cdot,\underline{\Gamma}''_B(C,\mathscr{P},3))$ to k, which will vanish on the coboundaries and hence induce the map in the cohomology group that we are looking for. Suppose that we start with a cocyle as above, given by the data:

(i) $\gamma_i\in\Lambda^3(\mathcal{O}_C,\underline{\mathscr{P}})^\times(U_i)$, for all $i\in I$.

(ii) $\varepsilon_{i,c}\in B_2(k(\mathfrak{c}))$ for every $c\in U_i$ all but finitely many of which are 0, for all $i\in I$

(iii) $\beta_{ij}\in(B_2(\mathcal{O}_C,\underline{\mathscr{P}})\otimes(\mathcal{O}_C,\underline{\mathscr{P}})^\times)(U_{ij})$, for all i, $j\in I$

These data are supposed to satisfy the following properties:

(i) $\delta(\beta_{ij})=\gamma_j|_{U_{ij}}-\gamma_i|_{U_{ij}}$,

(ii) $res_{\mathfrak{c}}(\beta_{ij})=\varepsilon_{j,c}-\varepsilon_{i,c}$, for $c\in U_{ij}$,

(iii) $\beta_{jk}|_{U_{ijk}} - \beta_{ik}|_{U_{ijk}} + \beta_{ij}|_{U_{ijk}} = 0$.

We will construct the image of the above element by making several choices and then proving that the construction is independent of all the choices.

(i) Let $\tilde{\mathcal{A}}_\eta/k_\infty$ be a smooth lifting of $\mathcal{O}_{C,\eta}$ and for every $c \in |C|$, let $\tilde{\mathcal{A}}_c/k_\infty$ be a smooth lifting of the completion $\hat{\mathcal{O}}_{C,c}$ of the local ring of C at c, together with a smooth lifting $\tilde{\mathfrak{c}}$ of $\mathfrak{c}$ as in the previous section. Moreover, choose:

(ii) an arbitrary $i \in I$ and for each c choose a $j_c \in I$ such that $c \in U_{j_c}$

(iii) an arbitrary lifting $\tilde{\gamma}_{i\eta} \in \Lambda^3\tilde{\mathcal{A}}_\eta^\times$ of the germ $\gamma_{i\eta} \in \Lambda^3\mathcal{O}_{C,\eta}^\times$ of γ_i at the generic point η

(iii) a good lifting $\tilde{\gamma}_{j_c} \in \Lambda^3(\tilde{\mathcal{A}}_c, \tilde{\mathfrak{c}})^\times$ of the image $\hat{\gamma}_{j_c,c}$ of γ_{j_c} in $\Lambda^3(\hat{\mathcal{O}}_{C,c}, \mathfrak{c})^\times$, for every $c \in |C|$,

(iv) an arbitrary lifting $\tilde{\beta}_{j_ci,\eta} \in B_2(\tilde{\mathcal{A}}_\eta) \otimes \tilde{\mathcal{A}}_\eta$ of the image $\beta_{j_ci,\eta} \in B_2(\mathcal{O}_{C,\eta}) \otimes \mathcal{O}_{C,\eta}^\times$ of β_{j_ci}, for every $c \in |C|$.

Note that it does not make sense to require that $\tilde{\gamma}_{i\eta}$ be a good lifting since in this context there is no a fixed specialization of the generic point. Similarly, we cannot require that $\tilde{\beta}_{j_ci,\eta}$ be a good lifting, since we know that $\delta(\tilde{\beta}_{j_ci,\eta})$ is a lifting of $\delta(\beta_{j_ci,\eta}) = \gamma_i - \gamma_{j_c}$ and even this last expression need not be good at c as γ_i need not be good at c.

We define the value of the regulator $\rho_{m,r}$ on the above element by the expression

$$\sum_{c\in|C|} \mathrm{Tr}_k\big(\ell_{m,r}(res_{\tilde{\mathfrak{c}}}\tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + res_c\omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_ci,\eta}), \tilde{\gamma}_{j_c})\big). \tag{8.1.1}$$

Let us first explain what we mean by the above expression. Since $\tilde{\gamma}_{j_c}$ is $\tilde{\mathfrak{c}}$-good, the residue $res_{\tilde{\mathfrak{c}}}\tilde{\gamma}_{j_c}$ along $\tilde{\mathfrak{c}}$ is defined as an element of $\Lambda^2k(\tilde{\mathfrak{c}})^\times$. The étaleness of $\tilde{\mathfrak{c}}$ over k_∞, implies that we have a canonical isomorphism $k(\tilde{\mathfrak{c}}) \simeq k(c)_\infty$ of k_∞-algebras. Using this isomorphism and the map $\ell_{m,r} : \Lambda^2k(c)_\infty^\times \to k(c)$ in Definition 3.0.2, we obtain $\ell_{m,r}(res_{\tilde{\mathfrak{c}}}\tilde{\gamma}_{j_c}) \in k(c)$. For the second term, note that, as above, there is a canonical isomorphism $k(\mathfrak{c}) \simeq k(c)_m$ of k_m algebras using which we can view $\varepsilon_{j_c,c} \in B_2(k(c)_m)$. Applying $\ell i_{m,r} : B_2(k(c)_m) \to k(c)$ to this element gives $\ell i_{m,r}(\varepsilon_{j_c,c}) \in k(c)$. For the last term, note that $\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_ci,\eta})$ is a lifting of $\gamma_{i\eta} - \delta(\beta_{j_ci,\eta}) = \gamma_{j_c}$ to $\Lambda^3\tilde{\mathcal{A}}_\eta^\times$ and so is $\tilde{\gamma}_{j_c}$ a lifting of γ_{j_c} to $\Lambda^3\tilde{\mathcal{A}}_c^\times$. Using the theory of §6.5, we see that the last term $res_c\omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_ci,\eta}), \tilde{\gamma}_{j_c}) \in k(c)$ is unambiguously defined. Letting Tr_k denote the normalized trace to k, the summands above are defined.

In order to show that the sum makes sense, we also need to show that the sum is finite. Below we will show that the sum is independent of all the choices, therefore it will be enough to show that the sum is finite for a particular choice. First by shrinking U_i if necessary, and choosing a refinement of the cover, we will assume that $\gamma_i \in \Lambda^3\mathcal{O}_C^\times(U_i)$. Similarly, by shrinking U_i even further, we will assume that the lifting $\tilde{\gamma}_i$ is good on U_i. Therefore, for $c \in U_i$, we can choose $j_c = i$ and $\tilde{\gamma}_{j_c} = \tilde{\gamma}_i$. Since for these c, $\beta_{j_ci} = 0$ we can choose $\tilde{\beta}_{j_ci} = 0$. In order to show that the sum in (8.1.1) is finite, we can concentrate on $c \in U_i$, as $|C| \setminus |U_i|$ is finite. For $c \in U_i$, $res_c\tilde{\gamma}_{j_c} = res_c\tilde{\gamma}_i = 0$, since γ_i is invertible on U_i by assumption. Also for the residues we have $res_c\omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_ci,\eta}), \tilde{\gamma}_{j_c}) = res_c\omega_{m,r}(\tilde{\gamma}_i, \tilde{\gamma}_i) = 0$ since $i = j_c$, $\tilde{\gamma}_{j_c} = \tilde{\gamma}_i$ and $\tilde{\beta}_{j_ci} = 0$. Therefore the summand, for $c \in U_i$, is equal to $\mathrm{Tr}_k(-\ell i_{m,r}(\varepsilon_{j_c,c})) = \mathrm{Tr}_k(-\ell i_{m,r}(\varepsilon_{i,c}))$. Since $\varepsilon_{i,c} = 0$, for all but a finite number of $c \in U_i$, we are done.

We now show that the expression makes sense and is independent of all the choices. Note that there are many of them.

Theorem 8.1.1. *For every $m < r < 2m$, the above formula (8.1.1) gives a well-defined regulator map $\rho_{m,r} : \check{\mathrm{Z}}^3(U_\bullet, \underline{\Gamma}''_B(C, \mathscr{P}, 3)) \to k$, independent of all the choices. This map vanishes on the coboundaries and hence induces the regulator map*

$$\rho_{m,r} : \mathrm{H}^3_B(C, \mathbb{Q}(3)) \to k$$

of $\star$-weight r.

Specializing to the case when C is the projective line $\mathbb{P}^1_{k_m}$, with coordinate function z, we fix an $a \in k_m^\flat$. If we choose $\mathscr{P}$ such that that z, $1-z$ and $z-a$ are all good with respect to $\mathscr{P}$, then $(1-z) \wedge z \wedge (z-a) \in \Gamma(\Lambda^3(\mathcal{O}_{\mathbb{P}^1}, \underline{\mathscr{P}})^\times)$ and

$$\rho_{m,r}((1-z) \wedge z \wedge (z-a)) = \ell i_{m,r}([a]).$$

Proof. We first show the independence of the definition from the various choices. For readability, we separate these into parts.

Independence of the choice of j_c *and the liftings* $\tilde{\beta}_{j_c i}$ *and* $\tilde{\gamma}_{j_c}$. Suppose that we choose a different j'_c with $c \in U_{j'_c}$; a different lifting $\tilde{\mathcal{A}}'_c$ of $\hat{\mathcal{O}}_{C,c}$, together with $\tilde{\mathfrak{c}}'$ as above; a $\tilde{\mathfrak{c}}'$-good lifting $\tilde{\gamma}'_{j'_c}$ of $\gamma_{j'_c}$ to $\tilde{\mathcal{A}}'_c$; and a lifting $\tilde{\beta}'_{j'_c i}$ of $\beta_{j'_c i}$ to $\tilde{\mathcal{A}}_\eta$. Since $\tilde{\mathcal{A}}_c \simeq k(c)[[\tilde{\mathfrak{s}}]]_\infty$, where $\tilde{\mathfrak{s}}$ is a choice of a uniformizer associated to $\tilde{\mathfrak{c}}$ and similarly for $\tilde{\mathcal{A}}'_c$, we choose and fix a k_∞-algebra isomorphism between $\tilde{\mathcal{A}}_c$ and $\tilde{\mathcal{A}}'_c$ which is identity modulo (t^m) and which sends $\tilde{\mathfrak{s}}$ to $\tilde{\mathfrak{s}}'$. This last condition is possible to impose since both $\tilde{\mathfrak{s}}$ and $\tilde{\mathfrak{s}}'$ lift $\mathfrak{s}$ by assumption. Below we identify these two algebras using this isomorphism.

We need to compare the two expressions

$$\ell_{m,r}(res_{\tilde{\mathfrak{c}}}\tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + res_c\omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i}), \tilde{\gamma}_{j_c}) \tag{8.1.2}$$

and

$$\ell_{m,r}(res_{\tilde{\mathfrak{c}}'}\tilde{\gamma}'_{j'_c}) - \ell i_{m,r}(\varepsilon_{j'_c,c}) + res_c\omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}'_{j'_c i}), \tilde{\gamma}'_{j'_c}). \tag{8.1.3}$$

By linearity we have

$$res_c\omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}'_{j'_c i}), \tilde{\gamma}'_{j'_c}) - res_c\omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i}), \tilde{\gamma}_{j_c}) = res_c\omega_{m,r}(\tilde{\gamma}_{j_c} - \tilde{\gamma}'_{j'_c}, \delta(\tilde{\beta}'_{j'_c i} - \tilde{\beta}_{j_c i})).$$

Let $\tilde{\beta}_{j'_c j_c}$ be a $\tilde{\mathfrak{c}}$-good lifting of $\beta_{j'_c j_c}$ to $\tilde{\mathcal{A}}_c$. Since $\beta_{j'_c j_c}$ itself is $\mathfrak{c}$-good such a lifting exists. We have the identity $\beta_{j'_c j_c} = \beta_{j'_c i} - \beta_{j_c i}$ on $U_{i j_c j'_c}$, which might not contain c, but does of course contain the generic point η. We deduce that $\tilde{\beta}_{j'_c j_c,\eta}$ and $\tilde{\beta}'_{j'_c i,\eta} - \tilde{\beta}_{j_c i,\eta}$ have the same reduction $\beta_{j'_c j_c,\eta}$. Now by Corollary 6.5.3, we conclude that

$$res_c\omega_{m,r}(\delta(\tilde{\beta}_{j'_c j_c,\eta}), \delta(\tilde{\beta}'_{j'_c i,\eta} - \tilde{\beta}_{j_c i,\eta})) = 0.$$

This implies that, using transitivity and linearity, we have:

$$res_c\omega_{m,r}(\tilde{\gamma}_{j_c} - \tilde{\gamma}'_{j'_c}, \delta(\tilde{\beta}'_{j'_c i} - \tilde{\beta}_{j_c i})) = res_c\omega_{m,r}(\tilde{\gamma}_{j_c} - \tilde{\gamma}'_{j'_c}, \delta(\tilde{\beta}_{j'_c j_c,\eta})) = res_c\omega_{m,r}(\tilde{\gamma}_{j_c} - \delta(\tilde{\beta}_{j'_c j_c,\eta}), \tilde{\gamma}'_{j'_c}).$$

In this expression, $\tilde{\gamma}_{j_c} - \delta(\tilde{\beta}_{j'_c j_c})$ is a $\tilde{\mathfrak{c}}$-good lifting to $\tilde{\mathcal{A}}_c$ and $\tilde{\gamma}'_{j'_c}$ is a $\tilde{\mathfrak{c}}'$-good lifting to $\tilde{\mathcal{A}}'_c$. Then Proposition 7.0.3 implies that

$$res_c\omega_{m,r}(\tilde{\gamma}_{j_c} - \delta(\tilde{\beta}_{j'_c j_c}), \tilde{\gamma}'_{j'_c}) = \ell_{m,r}(res_{\tilde{\mathfrak{c}}}(\tilde{\gamma}_{j_c} - \delta(\tilde{\beta}_{j'_c j_c}))) - \ell_{m,r}(res_{\tilde{\mathfrak{c}}'}(\tilde{\gamma}'_{j'_c})). \tag{8.1.4}$$

On the other hand,

$$\ell_{m,r}(res_{\tilde{\mathfrak{c}}}(\delta(\tilde{\beta}_{j'_c j_c}))) = \ell_{m,r}(\delta(res_{\tilde{\mathfrak{c}}}(\tilde{\beta}_{j'_c j_c}))) = \ell i_{m,r}(res_c(\beta_{j'_c j_c})),$$

by the definition of $\ell i_{m,r}$. Since by assumption $res_c(\beta_{j'_c j_c}) = \varepsilon_{j_c,c} - \varepsilon_{j'_c,c}$, we can rewrite the right hand side of (8.1.4) as

$$\ell_{m,r}(res_{\tilde{\mathfrak{c}}}(\tilde{\gamma}_{j_c})) - \ell_{m,r}(res_{\tilde{\mathfrak{c}}'}(\tilde{\gamma}'_{j'_c})) - \ell i_{m,r}(\varepsilon_{j_c,c}) + \ell i_{m,r}(\varepsilon_{j'_c,c}).$$

Combining all of the above, we see that the last expression is equal to the difference

$$res_c\omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}'_{j'_c i}), \tilde{\gamma}'_{j'_c}) - res_c\omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i}), \tilde{\gamma}_{j_c}),$$

which implies the equality of the two expressions (8.1.2) and (8.1.3) and thus proves the independence we were looking for.

Independence of the choice of i *and the liftings* $\tilde{\gamma}_{i\eta}$ *and* $\tilde{\beta}_{j_c i}$. Let us choose an i', a lifting $\tilde{\mathcal{A}}'_\eta$ of $\mathcal{O}_{C,\eta}$ and liftings $\tilde{\gamma}'_{i'\eta}$ and $\tilde{\beta}'_{j_c i'}$ to $\tilde{\mathcal{A}}'_\eta$, for each $c \in |C|$.

We need to compare

$$\sum_{c\in|C|} \mathrm{Tr}_k\big(\ell_{m,r}(res_{\tilde{\mathfrak{c}}}\tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + res_c\omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i}), \tilde{\gamma}_{j_c})\big) \tag{8.1.5}$$

and

$$\sum_{c\in|C|} \mathrm{Tr}_k\big(\ell_{m,r}(res_{\tilde{\mathfrak{c}}}\tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + res_c\omega_{m,r}(\tilde{\gamma}'_{i'\eta} - \delta(\tilde{\beta}'_{j_c i'}), \tilde{\gamma}_{j_c})\big). \tag{8.1.6}$$

The difference between the above expressions is

$$\sum_{c\in|C|}\mathrm{Tr}_k res_c\omega_{m,r}(\tilde{\gamma}'_{i'\eta}-\delta(\tilde{\beta}'_{j_ci'}),\tilde{\gamma}_{i\eta}-\delta(\tilde{\beta}_{j_ci})).$$

Choosing an isomorphism $\tilde{\mathcal{A}}_\eta\simeq\tilde{\mathcal{A}}'_\eta$ of k_∞-algebras which lifts the given one modulo (t^m), we identify $\tilde{\mathcal{A}}_\eta$ and $\tilde{\mathcal{A}}'_\eta$. The above sum can then be rewritten as:

$$\sum_{c\in|C|}\mathrm{Tr}_k res_c\omega_{m,r}(\tilde{\gamma}'_{i'\eta}-\tilde{\gamma}_{i\eta},\delta(\tilde{\beta}'_{j_ci'}-\tilde{\beta}_{j_ci})).$$

As in the above argument since $\tilde{\beta}_{ii'}$ has the same reduction modulo (t^m) as $\tilde{\beta}'_{j_ci'}-\tilde{\beta}_{j_ci}$ for any j_c, we have $res_c\omega_{m,r}(\delta(\tilde{\beta}'_{j_ci'}-\tilde{\beta}_{j_ci}),\delta(\tilde{\beta}_{ii'}))=0$ by Corollary 6.5.3. So we can rewrite the above sum as:

$$\sum_{c\in|C|}\mathrm{Tr}_k res_c\omega_{m,r}(\tilde{\gamma}'_{i'\eta}-\tilde{\gamma}_{i\eta},\delta(\tilde{\beta}_{ii'})).$$

Choosing a splitting of $\tilde{\mathcal{A}}_\eta$, we identify this algebra with $(\underline{\tilde{\mathcal{A}}}_\eta)_\infty=(\mathcal{O}_{\underline{C},\eta})_\infty$. Using this identification, the last expression is the sum of residues of the meromorphic 1-form $\Omega_{m,r}(\tilde{\gamma}'_{i'\eta}-\tilde{\gamma}_{i\eta}-\delta(\tilde{\beta}_{ii'}))$ on $\underline{C}$ and therefore is equal to 0.

Vanishing on coboundaries. Suppose that we start with sections

$$\alpha_i\in(B_2(\mathcal{O}_C,\underline{\mathscr{P}})\otimes(\mathcal{O}_C,\underline{\mathscr{P}})^\times)(U_i),$$

for all $i\in I$. Then we need to show that the value of the regulator on the data

$$(\{\gamma_i\}_{i\in I},\{\varepsilon_{i,c}|i\in I,\,c\in U_i\},\{\beta_{ij}\}_{i,j\in I})$$

is 0. Here $\gamma_i:=\delta(\alpha_i)$, $\varepsilon_{i,c}:=res_{\mathfrak{c}}(\alpha_i)$ and $\beta_{ij}:=\alpha_j|_{U_{ij}}-\alpha_i|_{U_{ij}}$.

We fix an $i\in I$ and $j_c\in I$, with $c\in U_{j_c}$, for every $c\in|C|$; and local and generic liftings $\tilde{\mathcal{A}}_c$, and $\tilde{\mathcal{A}}_\eta$ of the curve, as above, together with liftings $\tilde{\mathfrak{c}}$ of $\mathfrak{c}$ to $\tilde{\mathcal{A}}_c$. We need to choose liftings of the data in order to compute the value of the regulator on the above element.

We choose a lifting $\tilde{\alpha}_{i\eta}$ of $\alpha_{i\eta}$ to $\tilde{\mathcal{A}}_\eta$ and let $\tilde{\gamma}_{i\eta}:=\delta(\tilde{\alpha}_{i\eta})$. For each $c\in|C|$, we choose a $\tilde{\mathfrak{c}}$-good lifting $\tilde{\alpha}_{j_c}$ of α_{j_c} and let $\tilde{\gamma}_{j_c}:=\delta(\tilde{\alpha}_{j_c})$. Finally, we choose an arbitrary lifting $\tilde{\alpha}_{j_c\eta}$ of $\alpha_{j_c\eta}$ to $\tilde{\mathcal{A}}_\eta$, for every $c\in|C|$, and let $\tilde{\beta}_{j_ci,\eta}:=\tilde{\alpha}_{i\eta}-\tilde{\alpha}_{j_c\eta}$. Then the value of the regulator (8.1.1) is the sum of traces of the terms:

$$\begin{aligned}&\ell_{m,r}(res_{\tilde{\mathfrak{c}}}\tilde{\gamma}_{j_c})-\ell i_{m,r}(\varepsilon_{j_c,c})+res_c\omega_{m,r}(\tilde{\gamma}_{i\eta}-\delta(\tilde{\beta}_{j_ci,\eta}),\tilde{\gamma}_{j_c})\\=&\ell_{m,r}(res_{\tilde{\mathfrak{c}}}\delta(\tilde{\alpha}_{j_c}))-\ell i_{m,r}(\varepsilon_{j_c,c})+res_c\omega_{m,r}(\delta(\tilde{\alpha}_{j_c\eta}),\delta(\tilde{\alpha}_{j_c}))\\=&\ell_{m,r}(res_{\tilde{\mathfrak{c}}}\delta(\tilde{\alpha}_{j_c}))-\ell i_{m,r}(\varepsilon_{j_c,c})\end{aligned}$$

by Corollary 6.5.3. Since $res_{\tilde{\mathfrak{c}}}\delta(\tilde{\alpha}_{j_c})=\delta(res_{\tilde{\mathfrak{c}}}\tilde{\alpha}_{j_c})$, we have $\ell_{m,r}(res_{\tilde{\mathfrak{c}}}\delta(\tilde{\alpha}_{j_c}))=\ell_{m,r}(\delta(res_{\tilde{\mathfrak{c}}}\tilde{\alpha}_{j_c}))$. By the definition of $\ell i_{m,r}$, we have $\ell_{m,r}(\delta(res_{\tilde{\mathfrak{c}}}\tilde{\alpha}_{j_c}))=\ell i_{m,r}(res_{\mathfrak{c}}\alpha_{j_c})=\ell i_{m,r}(\varepsilon_{j_c,c})$. This implies that all the summands in the formula for the regulator (8.1.1) are 0 finishing the proof of the first part of the theorem.

A more general version of the computation for $\mathbb{P}^1_{k_m}$ will be done in §9.2. □

8.2. **Infinitesimal Chow Dilogarithm.** Specializing the above construction to global sections of $\Lambda^3(\mathcal{O}_C,\underline{\mathscr{P}})^\times$ gives us the generalization of the infinitesimal Chow dilogarithm in [21] to higher moduli.

Let us denote the global sections $\Gamma(\underline{C},(\mathcal{O}_C,\underline{\mathscr{P}})^\times)$ of $(\mathcal{O}_C,\underline{\mathscr{P}})^\times$ by $k(C,\underline{\mathscr{P}})^\times$. Suppose that we start with $\gamma\in\Lambda^3k(C,\underline{\mathscr{P}})^\times$. Specializing the construction in the previous section, we have $\rho_{m,r}(\gamma)\in k$, which can be computed as follows.

Choose a lifting $\tilde{\mathcal{A}}_\eta/k_\infty$ of $\mathcal{O}_{C,\eta}$ and local liftings $\tilde{\mathcal{A}}_c$ of $\hat{\mathcal{O}}_{C,c}$, for every $c\in|C|$, together with liftings $\tilde{\mathfrak{c}}$ of $\mathfrak{c}$. Choose an arbitrary lifting $\tilde{\gamma}_\eta$ of γ_η to $\tilde{\mathcal{A}}_\eta$ and $\tilde{\mathfrak{c}}$-good liftings $\tilde{\gamma}_c$ of the germ of γ at c to $\tilde{\mathcal{A}}_c$, for every $c\in|C|$.

By the definition in the previous section, we have

$$\rho_{m,r}(\gamma) := \sum_{c \in |C|} \mathrm{Tr}_k(\ell_{m,r}(res_{\tilde{\mathfrak{c}}}(\tilde{\gamma}_c)) + res_c \omega_{m,r}(\tilde{\gamma}_\eta, \tilde{\gamma}_c)), \tag{8.2.1}$$

for every $m < r < 2m$.

Corollary 8.2.1. *The definition in (8.2.1) of the infinitesimal Chow dilogarithm of modulus m and $\star$-weight r gives a map*

$$\rho_{m,r} : \Lambda^3 k(C, \underline{\mathscr{P}})^\times \to k,$$

independent of all the choices and generalizing the construction in [21] *to arbitrary m and r with $2 \leq m < r < 2m$.*

9. Applications and examples

9.1. **Strong reciprocity conjecture.** The infinitesimal Chow dilogarithm can be used to give a proof of an infinitesimal version of Goncharov's strong reciprocity conjecture for the curve C/k_m, exactly as in [21, Theorem 3.4.4]. In this section, in addition to our previous hypotheses, we assume that k is algebraically closed.

Taking the infinitesimal part of the sum of the residues at all closed points of C, we have a map:

$$res_C : \Lambda^3 k(C, \underline{\mathscr{P}})^\times \to (\Lambda^2 k_m^\times)^\circ.$$

Similarly, letting $B_2(k(C, \underline{\mathscr{P}}))$ denote the set of global sections of $B_2(\mathcal{O}_C, \underline{\mathscr{P}})$, we have a map

$$res_C : B_2(k(C, \underline{\mathscr{P}})) \otimes k(C, \underline{\mathscr{P}})^\times \to B_2(k_m)^\circ.$$

The explicit version of the strong reciprocity conjecture expresses both of these maps in terms of a single map h :

Theorem 9.1.1. *There is a map $h : \Lambda^3 k(C, \underline{\mathscr{P}})^\times \to B_2(k_m)^\circ$, which makes the diagram*

$$\begin{array}{ccc} B_2(k(C, \underline{\mathscr{P}})) \otimes k(C, \underline{\mathscr{P}})^\times & \xrightarrow{\delta} & \Lambda^3 k(C, \underline{\mathscr{P}})^\times \\ \downarrow{\scriptstyle res_C} & \overset{h}{\swarrow} & \downarrow{\scriptstyle res_C} \\ B_2(k_m)^\circ & \xrightarrow{\delta^\circ} & (\Lambda^2 k_m^\times)^\circ \end{array}$$

commute and has the property that $h(k_m^\times \wedge \Lambda^2 k(C, \underline{\mathscr{P}})^\times) = 0$.

Proof. The proof, using the maps $\rho_{m,r}$ constructed above, is exactly the same as that of [21, Theorem 3.4.4] and is omitted. □

The theorem above, in essence, states that the residue map from the Bloch complex of weight 3 on C to the Bloch complex of weight 2 on k_m is homotopic to 0, cf. [21, §3.4].

9.2. **The special case of the projective line.** As a first example, let us look at the infinitesimal Chow dilogarithm in the case of the projective line $\mathbb{P}^1$ over k_m with k algebraically closed.

For each point $c \in \mathbb{P}^1_k$ let us choose a smooth lifting $\mathfrak{c} \in \mathbb{P}^1(k_m)$. Considering a lifting as a map $\mathrm{Spec}(k_m) \to \mathbb{P}^1_{k_m}$, if the projection $\mathrm{Spec}(k_m) \to \mathbb{P}^1_k$ factors through the structure map $\mathrm{Spec}(k_m) \to \mathrm{Spec}(k)$, we call that lifting a constant lifting. In the following, we will always choose the constant liftings of the points 0, 1 and ∞, for the other points in $\mathbb{P}^1_k$ the choices will be arbitrary. We denote the set of these liftings by $\mathscr{P}$ as usual.

Letting $a \in \mathscr{P}$ be the chosen lifting of an element in $k^\flat$, the element $(1-z) \wedge z \wedge (z-a)$ satisfies our goodness hypothesis. We will compute the value of $\rho_{m,r}$ on this element.

We will use the formula (1.3.1) directly. In order to do this first let us choose a set $\tilde{\mathscr{P}}$ of liftings to $\mathbb{P}^1_{k_\infty}$ of elements in $\mathscr{P}$. Again for the elements 0, 1, and ∞, we choose the constant liftings. Let us denote the lifting of $a \in \mathscr{P}$ by $\tilde{a} \in \tilde{\mathscr{P}}$. Then the functions z, $1-z$, and $z - \tilde{a}$ are liftings of z, $1-z$, and $z-a$ to functions on $\mathbb{P}^1_{k_\infty}$, which are good with respect to $\tilde{\mathscr{P}}$. Using the definition of $\rho_{m,r}$, we obtain

$$\rho_{m,r}((1-z)\wedge z\wedge(z-a))=\sum_{\tilde{\mathfrak{c}}\in\tilde{\mathscr{P}}}\ell_{m,r}(res_{\tilde{\mathfrak{c}}}((1-z)\wedge z\wedge(z-\tilde{a}))).$$

The only contribution to the sum above comes from the residue at $\tilde{a}$. Therefore the last expression is equal to $\ell_{m,r}((1-\tilde{a})\wedge\tilde{a})$. Using the definition of $\ell i_{m,r}$ we can rewrite this as $\ell_{m,r}((1-\tilde{a})\wedge\tilde{a})=\ell_{m,r}(\delta[\tilde{a}])=\ell i_{m,r}([a])$.

Let f, g and h be arbitrary functions on $\mathbb{P}^1_{k_m}$ which are good with respect to $\mathscr{P}$. Then we can write

$$f=\lambda\prod_{1\le i\le l}(z-\alpha_i)^{\delta_i},\quad g=\mu\prod_{1\le j\le m}(z-\beta_j)^{\varepsilon_j},\quad\text{and},\quad h=\nu\prod_{1\le k\le n}(z-\gamma_k)^{\eta_k},$$

for some α_i, β_j, $\gamma_k\in\mathscr{P}$, λ, μ, $\nu\in k_m^\times$ and δ_i, ε_j, $\eta_k\in\mathbb{Z}$. Using exactly the same argument at the end of §2.2, we find that $\rho_{m,r}(f\wedge g\wedge h)$ is equal to

$$\sum_{i,j,k}\delta_i\varepsilon_j\eta_k\cdot\ell i_{m,r}([\frac{\gamma_k-\beta_j}{\alpha_i-\beta_j}]) \tag{9.2.1}$$

where the summation is on $1\le i\le l$, $1\le j\le m$ and $1\le k\le n$. In this notation, we use the convention that $\ell i_{m,r}([\frac{\gamma_k-\beta_j}{\alpha_i-\beta_j}])=0$, if at least two of α_i, β_j and γ_k are the same. Note that because of the goodness with respect to $\mathscr{P}$ hypothesis, if α_i, β_j and γ_k are pairwise distinct then their reductions to k have to be distinct as well. This, then, implies that $\frac{\gamma_k-\beta_j}{\alpha_i-\beta_j}\in k_m^\flat$ and the expression (9.2.1) is well-defined.

9.3. **The special case of elliptic curves.** As a second example, we will consider the infinitesimal Chow dilogarithm in the case of an elliptic curve. Again, for simplicity, we assume that k is algebraically closed. Suppose that E/k_m is an elliptic curve. Suppose that $E\subseteq\mathbb{P}^2_{k_m}$ is given by a Weierstrass equation $y^2=x^3+Ax+B$, with $A,B\in k_m$, in the affine coordinates with $x=X/Z$ and $y=Y/Z$, where X,Y, and Z are the homogeneous coordinates on $\mathbb{P}^2$. Suppose further that we have a lifting $\tilde{E}\subseteq\mathbb{P}^2_{k_\infty}$ which is also given by a Weierstrass equation $y^2=x^3+\tilde{A}x+\tilde{B}$, with $\tilde{A},\tilde{B}\in k_\infty$. Note that these hypotheses are satisfied when E is constant curve, in other words $E=\underline{E}\times_k k_m$, for some elliptic curve $\underline{E}/k$. In this case, we can of course choose the lifting as $\underline{E}\times_k k_\infty$. In the following, we will not assume that our curve is a constant curve. Suppose we fix a choice of smooth liftings $\mathscr{P}$ as above for each point in $E(k)$ such that the lifting of the origin in $E(k)$ is the origin in $E(k_m)$.

Let l_0 be the line which intersects the curve at the origin O with multiplicity 3. It is the line given by the equation $Z=0$. For each $1\le i\le 3$ let l_i be the line given by $a_iX+b_iY+c_iZ=0$, with a_i, b_i, $c_i\in k_m$. Denote the intersection points of the line l_i with the elliptic curve by α_{i1},α_{i2} and α_{i3}. Note that the group law on E gives that $\alpha_{i1}+\alpha_{i2}+\alpha_{i3}=0$. Suppose that the intersection points α_{i1},α_{i2}, α_{i3} lie in the chosen set of liftings $\mathscr{P}$. Let f_i denote the function on E given by l_i/l_0. For a generic choice of the lines, let us compute $\rho_{m,r}(f_1\wedge f_2\wedge f_3)$.

Let $\tilde{l}_0$ be the line in $\mathbb{P}^2_{k_\infty}$ given by $Z=0$ and $\tilde{l}_i$ be the line given by $\tilde{a}_iX+\tilde{b}_iY+\tilde{c}_iZ=0$ for some liftings $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_i\in k_\infty$ of a_i, b_i, $c_i\in k_m$. Then the functions $\tilde{f}_i:=\tilde{l}_i/\tilde{l}_0$ are liftings of f_i. If $\tilde{\alpha}_{i1}$, $\tilde{\alpha}_{i2}$, $\tilde{\alpha}_{i3}$ are the intersections of $\tilde{l}_i$ with $\tilde{E}$, then we can compute the above regulator as follows. Choose a smooth lifting to $\tilde{E}$ of each element in $\mathscr{P}$, such that:

(i) the origin in $\tilde{E}(k_\infty)$ is the lifting of the origin in $E(k_m)$

(ii) the elements $\tilde{\alpha}_{i1}$, $\tilde{\alpha}_{i2}$, $\tilde{\alpha}_{i3}$ are the liftings of α_{i1}, α_{i2}, α_{i3} for $1\le i\le 3$.

Denote the set of these liftings by $\tilde{\mathscr{P}}$. By our formula, we have

$$\rho_{m,r}(f_1\wedge f_2\wedge f_3)=\sum_{\tilde{\mathfrak{c}}\in\tilde{\mathscr{P}}}\ell_{m,r}(res_{\tilde{\mathfrak{c}}}(\tilde{f}_1\wedge\tilde{f}_2\wedge\tilde{f}_3)).$$

Since we assume that the lines are generic, there are no common zeros of the functions $\tilde{f}_i$. On the other hand,

$$res_O(\tilde{f}_1\wedge\tilde{f}_2\wedge\tilde{f}_3)=-3(\tilde{b}_2\wedge\tilde{b}_3-\tilde{b}_1\wedge\tilde{b}_3+\tilde{b}_1\wedge\tilde{b}_2).$$

Combining these, we obtain that the value $\rho_{m,r}(f_1 \wedge f_2 \wedge f_3)$ is equal to

$$\ell_{m,r}\Big(\sum_{\sigma \in C_3} (-1)^{|\sigma|}\big(-3\tilde{b}_{\sigma(2)} \wedge \tilde{b}_{\sigma(3)} + \sum_{1 \le j \le 3} \tilde{f}_{\sigma(2)}(\tilde{\alpha}_{\sigma(1)j}) \wedge \tilde{f}_{\sigma(3)}(\tilde{\alpha}_{\sigma(1)j})\big)\Big), \tag{9.3.1}$$

where C_3 is the subgroup of S_3 generated by the 3-cycle (123).

In fact, the above expression gives an explicit computation for $\rho_{m,r}(f_1 \wedge f_2 \wedge f_3)$ for any generic choice of functions f_1, f_2, f_3 which are good with respect to $\mathscr{P}$. The group law on the elliptic curve has the property that a divisor of degree 0 is the divisor of a rational function if and only if the divisor adds to 0 under the group law. This implies that the functions f_i can be written as products of functions of the form l/l_0, where l is a line, and an element in $k_m^\times$. Since $\rho_{m,r}$ vanishes on elements of the form $\lambda \wedge f \wedge g$, with $\lambda \in k_m^\times$, using the additivity of $\rho_{m,r}$ we obtain the expression for $\rho_{m,r}(f_1 \wedge f_2 \wedge f_3)$ using the one for $\rho_{m,r}((l_1/l_0) \wedge (l_2/l_0) \wedge (l_3/l_0))$.

Our main theorem implies that the expression (9.3.1) is independent of the choices of the liftings of both E and the f_i's. When such global liftings of either the elliptic curve or the functions do not exist, we need to choose local liftings and add the defect for choosing different liftings on the intersections as in the formula (1.3.1).

9.4. **Invariants of cycles on k_m.** As in [21], this construction gives us an invariant of cycles. For a cycle of modulus m we expect the combination of $\rho_{m,r}$ for all $m < r < 2m$ to be a complete set of invariants for the rational equivalence class of a cycle. For the appropriate, yet to be defined, Chow group $\mathrm{CH}^2(k_m, 3)$, we expect that our regulators $\rho_{m,r}$ to give complete invariants for the infinitesimal part $\mathrm{CH}^2(k_m, 3)^\circ$. This section also generalizes Park's construction of regulators [15], where the case of $r = m+1$ is dealt with. Since this section is more or less a generalization of [21, §4], we do not go into the details and explain certain constructions in a slighty alternate way.

First, let us recall the definition of cubical higher Chow groups over a smooth k-scheme X/k [2]. Let $\square_k := \mathbb{P}^1_k \setminus \{1\}$ and $\square^n_k$ the n-fold product of $\square_k$ with itself over k, with the coordinate functions $y_1, \cdots, y_n$. For a smooth k-scheme X, we let $\square^n_X := X \times_k \square^n_k$. A codimension 1 face of $\square^n_X$ is a divisor F^a_i of the form $y_i = a$, for $1 \le i \le n$, and $a \in \{0, \infty\}$. A face of $\square^n_X$ is either the whole scheme $\square^n_X$ or an arbitrary intersection of codimension 1 faces. Let $\underline{z}^q(X, n)$ be the free abelian group on the set of codimension q, integral, closed subschemes $Z \subseteq \square^n_X$ which are *admissable*, i.e. which intersect each face properly on $\square^n_X$. For each codimension one face F^a_i, and irreducible $Z \in \underline{z}^q(X, n)$, we let $\partial^a_i(Z)$ be the cycle associated to the scheme $Z \cap F^a_i$. We let $\partial := \sum_{i=1}^n (-1)^n(\partial^\infty_i - \partial^0_i)$ on $\underline{z}^q(X, n)$, which gives a complex $(\underline{z}^q(X, \cdot), \partial)$. Dividing this complex by the subcomplex of degenerate cycles, we obtain Bloch's higher Chow group complex whose homology $\mathrm{CH}^q(X, n) := \mathrm{H}_n(z^q(X, \cdot))$ is the higher Chow group of X.

In order to work with a candidate for Chow groups of cycles on k_m, we need to work with cycles over k_∞ which have a certain finite reduction property. The following definitions are essentially from [21, §4.2]. Let $\overline{\square}_k := \mathbb{P}^1_k$, $\overline{\square}^n_k$, the n-fold product of $\overline{\square}_k$ with itself over k, and $\overline{\square}^n_{k_\infty} := \overline{\square}^n_k \times_k k_\infty$. We define a subcomplex $\underline{z}^q_f(k_\infty, \cdot) \subseteq \underline{z}^q(k_\infty, \cdot)$, as the subgroup generated by integral, closed subschemes $Z \subseteq \square^n_{k_\infty}$ which are admissible in the above sense and have *finite reduction*, i.e. $\overline{Z}$ intersects each $s \times \overline{F}$ properly on $\overline{\square}^n_{k_\infty}$, for every face F of $\square^n_{k_\infty}$. Here s denotes the closed point of the spectrum of k_∞ and for a subscheme $Y \subseteq \square^n_{k_\infty}$, $\overline{Y}$ denotes its closure in $\overline{\square}^n_{k_\infty}$. Modding out by degenerate cycles, we have a complex $z^q_f(k_\infty, \cdot)$.

Fix $2 \le m < r < 2m$. Let η denote the generic point of the spectrum of k_∞. An irreducible cycle p in $\underline{z}^2_f(k_\infty, 2)$ is given by a closed point p_η of $\square^2_\eta$ whose closure $\overline{p}$ in $\overline{\square}^2_{k_\infty}$ does not meet $(\{0,\infty\} \times \overline{\square}_{k_\infty}) \cup (\overline{\square}_{k_\infty} \times \{0, \infty\})$. Let $\tilde{p}$ denote the normalisation of $\overline{p}$ and T denote the underlying set of the closed fiber $\tilde{p} \times_{k_\infty} s$ of $\tilde{p}$. For every $s' \in T$, and $1 \le i$, define $\ell_{\tilde{p},s',i} : \hat{\mathcal{O}}^\times_{\tilde{p},s'} \to k(s')$ by the formula:

$$\ell_{\tilde{p},s',i}(y) := \frac{1}{i} res_{\tilde{p},s'} \frac{1}{t^i} d\log(y).$$

Let

$$l_{m,r}(p) := \sum_{s' \in T} \mathrm{Tr}_k \sum_{1 \le i \le r-m} i \cdot (\ell_{\tilde{p},s',r-i} \wedge \ell_{\tilde{p},s',i})(y_1 \wedge y_2). \tag{9.4.1}$$

Note the similarity with Definition 3.0.2.

Definition 9.4.1. *We define the regulator* $\rho_{m,r} : \underline{z}^2_f(k_\infty, 3) \to k$ *as the composition* $l_{m,r} \circ \partial$.

Exactly as in [21], one proves that the regulator above vanishes on boundaries and products, is alternating and has the same value on cycles which are congruent modulo (t^m). We state only this last property, which is the most important one, in detail.

Suppose that Z_i for $i = 1, 2$ are two irreducible cycles in $\underline{z}^2_f(k_\infty, 3)$. We say that Z_1 and Z_2 are equivalent modulo t^m if the following condition (M_m) holds:
(i) $\overline{Z}_i/k_\infty$ are smooth with $(\overline{Z}_i)_s \cup (\cup_{j,a}|\partial^a_j Z_i|)$ a strict normal crossings divisor on $\overline{Z}_i$.
and
(ii) $\overline{Z}_1|_{t^m} = \overline{Z}_2|_{t^m}$.
Then we have:

Theorem 9.4.2. *For* $m < r < 2m$, *we define a regulator* $\rho_{m,r} : \underline{z}^2_f(k_\infty, 3) \to k$. *If* Z_a, *for* $a \in k^\flat_\infty$ *is the dilogarithmic cycle given by the parametric equation* $(1 - z, z, z - a)$ *then*

$$\rho_{m,r}(Z_a) = \ell i_{m,r}([a]).$$

If $Z_i \in \underline{z}^2_f(k_\infty, 3)$, *for* $i = 1, 2$, *satisfy the condition* (M_m), *then they have the same infinitesimal regulator value:*

$$\rho_{m,r}(Z_1) = \rho_{m,r}(Z_2).$$

Proof. The second part of the proof is exactly as in [21] and is based on Corollary 8.2.1.

In order to compute $\rho_{m,r}(Z_a)$, we note that $\partial(Z_a) = (1 - a, a)$ and $\rho_{m,r}(Z_a) =$

$$(l_{m,r} \circ \partial)(Z_a) = l_{m,r}(1 - a, a) = \sum_{1 \le i \le r-m} i(\ell_{r-i} \wedge \ell_i)(1 - a, a) = (\ell_{m,r} \circ \delta)(a) = \ell i_{m,r}([a]).$$

□

As we remarked above, we expect the invariants $\rho_{m,r}$ for $m < r < 2m$ to give a full set of invariants in the infinitesimal part of a yet to be defined Chow group $\mathrm{CH}^2(k_m, 3)$.

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