A Short Course on Market Design and Matching

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Lecture 7: Matching with Contracts

A new concept

- Schools, hospitals, colleges, etc. rank the the other side of the market linearly. The preferences are simple in the sense that it is a linear order over the set of students.
- A more general class of preferences include concerns about the terms of the match as well.
- For example, certain contractual terms between a firm and a worker such as wage, the description of the job, working hours etc, or schools might have preferences over the set of subsets of students since students have different characteristics, such as ability, minority status, the socioeconomic status of the family, and the schools care about the composition of its students.

The model

D and H are sets of doctors and hospitals.

X is a set of contracts. Each contract $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$.

Each doctor can sign one contract (or no contract, $\emptyset$), but a hospital can sign multiple contracts.

P_i is a strict preference relation for $i \in D \cup H$.

$C_i(X')$ is the chosen set of contracts by i from $X' \subseteq X$

A set of contracts $X' \subseteq X$ is an allocation if each doctor signs at most one contract in X'.

The model

An allocation $X' \subseteq X$ is stable if

- Individual rationality: $C_D(X') = C_H(X') = X'$; and
- No blocking coalition: there exists no hospital h and set of contracts $Y \neq C_h(X')$ such that
  $Y = C_h(X' \cup Y) \subseteq C_D(X' \cup Y)$.

Substitutes

Definition

Contracts are **substitutes** (or, preferences are **substitutable**) for h if there are no contracts $x, z \in X$ and a set of contracts $Y \subseteq X$ such that $z \notin C_h(Y \cup \{z\})$ and $z \in C_h(Y \cup \{x, z\})$.

In other words, contracts are substitutes if the addition of a contract to the choice set never induces a hospital to take a contract it previously rejected.

A stable matching may not exist

- two hospitals h and h'; and two doctors d_1 and d_2.
- d_1 is a specialist and can also apply to a practitioner position.
- d_2 is a practitioner and cannot apply to a specialist position.
- h has up to two slots and it can hire a specialist only if a practitioner is already hired.
- h' has only one slot for a practitioner.
- for d_1, contract $x^\star$ is a specialist position at h; x and x' are practitioner positions at h and h', respectively.
- for d_1, contracts y and y' are practitioner positions at h and h', respectively.

A stable matching may not exist

- The preferences of the doctors are as follows:

  $\succ_{d_1}: x^\star \succ_{d_1} x' \succ_{d_1} x \succ_{d_1} \emptyset$
  $\succ_{d_2}: y' \succ_{d_2} y \succ_{d_2} \emptyset$

- The preferences of the hospitals are as follows:

  $P_h : \{x^\star, y\} \; P_h \; \{x\} \; P_h \; \{y\} \; P_h \; \emptyset$
  $P_{h'} : \{x'\} \; P_{h'} \; \{y'\} \; P_{h'} \; \emptyset$

A stable matching may not exist

- If a stable set of contracts S includes x', it cannot include contract y' and then the set $\{x^\star, y\}$ blocks S.
- If S includes y', it cannot include contract x' and then the set $\{x'\}$ blocks S.
- Thus, hospital h' has no contract in S. But then, the set $\{y'\}$ blocks S.
- Conclusion: there exists no stable matching.

Bilateral substitutes

Definition

Contracts are **bilateral substitutes** for hospital h if there do not exist contracts $x, z \in X$ and a set of contracts $Y \subseteq X$ such that $x_D, z_D \notin Y_D$, $z \notin C_h(Y \cup \{z\})$ and $z \in C_h(Y \cup \{x, z\})$

Bilateral substitutes is a weaker condition than substitutes in two ways.

- For a rejected contract z, only the sets of other contracts that do not involve z_D are considered.
- For a contract x that may be added to the set of contracts, only contracts with doctors not in Y_D are considered.

The generalized Deferred Acceptance

Step 1: One arbitrarily chosen doctor offers her first choice contract x_1. The hospital $h_1 = (x_1)_H$, **holds** the contract if it is acceptable and rejects it otherwise. Let $A_{h_1}(1) = \{x_1\}$, and for each $h \neq h_1$, $A_h(1) = \emptyset$. Let $A(1) = \bigcup_{h \in H} A_h(1)$.

Step $t \geqslant 2$: One of the doctors for whom no contract is currently held by a hospital offers the most preferred contract x_t, that has not been rejected in previous steps. Let $h_t = (x_t)_H$ hold $C_{h_t}(A_{h_t}(t-1) \cup \{x_t\})$ and reject all other contracts. Let $A_{h_t}(t) = \left(A_{h_t}(t-1) \cup \{x_t\}\right)$, and $A_h(t) = A_h(t-1)$ for $h \neq h_t$.

Slot-specific priorities

- a set of agents I
- a set of branches B
- a finite set of contracts X
- each $x \in X$ is between a agent $i(x) \in I$ and branch $b(x) \in S$
- each $b \in B$ has a set S_b of slots
- each slot can be assigned at most one contract in $X_b = \{x \in X : b(x) = b\}$.
- each slot $s \in S$ has (linear) priority order over X_b
- the choice set of a branch is determined by a precedence order of its slots

Slot-specific priorities

Theorem (Kominers and Sonmez, 2016)

The cumulative offers mechanism produces a stable outcome.

Theorem (Kominers and Sonmez, 2016)

The cumulative offers mechanism is strategy-proof.

Dynamic reserves

- Each school initially reserves a certain number of its slots for different privilege groups (student types).
- Each student is a member of at least one privilege group.
- Each school has a pre-specified order in which these different privilege groups are to be considered.
- Each student has a preference over school-privilege type pairs.
- Each school has a soft target distribution of its slots over privilege types.

Dynamic reserves

- If there is less demand from at least one privilege type, schools are given opportunity to utilize these vacant seats by transferring them over to other privilege groups.
- Schools might have preferences over how to redistribute these unassigned slots.
- Aygun and Turhan (2017) design choice functions for schools that allows them to transfer capacities from low-demand privilege types to high-demand privilege types.

Dynamic reserves

Theorem (Aygun and Turhan, 2017)

Under some mild assumptions, there exists a stable allocation.

Theorem (Aygun and Turhan, 2017)

Under some mild assumptions, the cumulative offers mechanism is strategy-proof for dynamic choice rules. allocation.

Matching under constraints

- Japan residency matching: rural hospitals cannot fill enough positions because too many doctors are allocated to metropolitan areas under the DA adopted in 2003.
- Government introduced a regional cap as a constraint.
- Same issue for the US market as well:
  "The United States suffers a terrible problem of maldistribution of physicians, with urban areas being relatively over-served and innercity and rural areas being relatively under-served. At present, although approximately 100 hospitals fill every residency position, there are over 100 hospitals that do not receive a single application. This maldistribution would only be worsened by the only rational alternative to the present matching program, the mirror-image program, which favors students."

Matching under constraints

- Each hospital belongs to a region.
- Each region has exogenous regional cap.
- The DA in use in Japan: Apply the DA with artificial capacities.

Matching under constraints

A new mechanism: flexible DA (FDA)

1. Apply the DA with artificial capacities.
2. With an exogenous order, hospitals take turns to (tentatively) accept favorite remaining applicants until (i) the regional cap becomes full or (ii) no doctor remains to be processed.

Matching under constraints

Theorem (Kamada and Kojima, 2015)

1. The FDA outcome is constrained Pareto efficient and stable.
2. The FDA is strategy-proof for doctors.
3. Every doctor weakly prefers FDA to JRMP.