A Short Course on Market Design and Matching

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Lecture 4: Assignment Problem

The scope

- How to allocate objects when
  - objects are heterogeneous
  - each agent receives at most one object
  - monetary transfers are not allowed
- On-campus housing, organ allocation, office allocation and (some) student placement problems.

The model with a social endowment

N: a set of agents

O: a set of objects

R_i: agent i's strict preferences over O

An **assignment** is a bijection $\mu : N \to O$.

A rule ϕ is a systematic procedure which selects an assignment for each problem $((R_i)_{i \in N})$.

The priority mechanism

Definition

A *priority mechanism* specifies an ordering of the agents and let the first agent receive her favourite object, the next agent his favourite objects among the remaining ones, and so on.

The priority mechanism

Theorem

The priority mechanism is Pareto-efficient and strategy-proof.

The random priority (RP) mechanism

Definition: The RP mechanism

An ordering of the agents is randomly drawn from the uniform distribution and then apply the priority mechanism for the ordering drawn.

The random priority (RP) mechanism

An example:

agents 1, 2: $a \succ b \succ c \succ d$
agents 3, 4: $b \succ a \succ d \succ c$

	a	b	c	d
1	5/12	1/12	5/12	1/12
2	5/12	1/12	5/12	1/12
3	1/12	5/12	1/12	5/12
4	1/12	5/12	1/12	5/12

Random mechanisms

- A **random consumption** is a probability distribution over O.
- Let $\triangle O$ denote the set of all random consumptions.
- A **lottery** is a probability distribution over deterministic assignments.
- Let $\triangle \mathcal{M}$ denote the set of all lotteries.
- Each lottery induces a **random assignment** $Q = [q_{ia}]_{i \in N, a \in O}$, where $q_{ia} \in [0, 1]$ is the probability that agent i receives object o.
- Let Q_i denote the resulting random consumption for agent i.
- A random assignment is represented as a bi-stochastic matrix.

Efficiency revisited

- A deterministic assignment is **Pareto efficient** if no other deterministic assignment makes each agent at least as well off and at least one agent better off.
- A lottery is **ex post efficient** if it gives positive weights only to *Pareto efficient* deterministic assignments.

A stronger efficiency concept

- For agent i, $\mathbf{Q_i}$ **stochastically dominates** $\mathbf{T_i}$, written as $\mathbf{Q_i}$ **sd**$(\mathbf{R_i})$ $\mathbf{T_i}$, if

$$\forall a \in O, \quad \sum_{a' \in U(R_i,a)} q_{ia'} \geq \sum_{a' \in U(R_i,a)} t_{ia'}.$$

- Given a pair of distinct random assignments Q, and T, **Q stochastically dominates T**, if, for each $i \in N$, $Q_i \; sd(R_i) \; T_i$.
- A random assignment is **sd-efficient** if it is not stochastically dominated by any other random assignment.
- A rule is **sd-efficient** if it always selects *sd-efficient* random assignments.

The RP is not sd-efficient

agents 1, 2: $a \succ b \succ c \succ d$
agents 3, 4: $b \succ a \succ d \succ c$

	a	b	c	d
1	5/12	1/12	5/12	1/12
2	5/12	1/12	5/12	1/12
3	1/12	5/12	1/12	5/12
4	1/12	5/12	1/12	5/12

	a	b	c	d
1	1/2	0	1/2	0
2	1/2	0	1/2	0
3	0	1/2	0	1/2
4	0	1/2	0	1/2

Fairness

- A matching μ satisfies **no-envy** is for each $i, j \in N$, $\mu(i)\ R_i\ \mu(j)$.
- A random assignment $Q = [q_{ia}]_{i \in N, a \in O}$ satisfies **sd-no-envy** if, for each $i, j \in N$, $Q_i\ sd(R_i)\ Q_j$.
- A rule satisfies **sd-no-envy** if it always selects *envy-free* random assignments.

The RP does not satisfy sd-no-envy

Theorem (Bogomolnaia and Moulin, 2001)

The RP does not satisfy sd-no-envy.

A new mechanism

Definition: Eating mechanism

Each object is imagined as being infinitely divisible. There is one unit of each object. A quantity of object *a* given to agent *i*, represents the probability with which agent *i* is assigned object *a*. For each agent *i*, let $\omega_i : [0, 1] \to \mathbb{R}_+$ be a function such that $\int_0^1 \omega_i(t)dt = 1$. The function ω_i is called the eating speed function of agent *i*, and $\omega_i(t)$ is the eating speed of agent *i* at time *t*. Given the profile of eating speeds $\omega = (\omega_i)_{i \in I}$ and the profile *R* of preferences, at time *t*, the *eating* algorithm lets agent *i* eat her favorite available object at the speed $\omega_i(t)$: if at time *t*, the objects $a, b, c \dots$ have been entirely eaten and objects $x, y, z \dots$ have not, she eats her favorite objects among $x, y, z \dots$ at the speed $\omega_i(t)$.

A new mechanism

Theorem (Bogomolnaia and Moulin, 2001)

For each profile of eating speeds, the eating algorithm gives an sd-efficient random assignment. Conversely, each sd-efficient random assignment can be obtained by the eating algorithm for some profile of eating speeds.

The Probabilistic Serial (PS) mechanism

The eating algorithm with uniform speeds define a rule, the **Probabilistic Serial (PS)** rule.

Theorem (Bogomolnaia and Moulin, 2001)

The PS rule satisfies sd-no-envy.

The RP vs PS

	sd-efficiency	sd-no-envy	sd-strategy-proofness
The RP	×	×	✓
The PS	✓	✓	×

An incompatibility result

Theorem (Bogomolnaia and Moulin, 2001)

No mechanism satisfies the following three properties together: sd-efficiency, sd-strategyproofness and equal treatment of equals.

The model with private endowments

N: a set of agents

O: a set of objects

P_i: agent i's strict preferences over O

An **endowment profile** is a bijection $\omega : N \to O$.

An **assignment** is a bijection $\mu : N \to O$.

A rule ϕ is a systematic procedure which selects an assignment for each problem $((P_i)_{i \in N}, \omega)$.

A central notion

Definition: The Core

A coalition $S \subseteq N$ blocks an assignment μ if the agents in S reject μ and reallocate their endowments among themselves such that at least one of them better off and none of them is worse off.

Gale's Top Trading Cycles (TTC) Algorithm

Step 1 Each agent "points to" the owner of his favorite object. Since there are finite number of agents, there is at least one cycle. Each agent in a cycle is assigned the object of the agent she points to and removed from the market with her assignment. If there is at least one remaining agent, proceed with the next step.

Step $k \geq 2$ Each remaining agent "points" to the owner of his favorite object among the remaining objects. Every agent in a cycle is assigned the object of the agent she points to and removed from the market with her assignment. If there is at least one remaining agent, proceed with the next step.

Existence of the core

Theorem (Shapley and Scarf, 1974)

There exists a core matching for any assignment problem with private endowments.

Uniqueness of the core

Theorem (Roth and Postlewaite, 1977)

The matching produced by Gale's TTC algorithm is the unique core matching.

Uniqueness of the core

Theorem (Ma, 1994)

The TTC is the only mechanism that is Pareto efficient, individually rational, and strategy-proof.

A social endowment and private endowments

- Some objects are private endowments and others are social endowment.
- This generalizes assignment problem both with and without private endowments.

Generalized Top Trading Cycles (TTC) Algorithm

Step 0 Fix a priority ordering of agents. Each object is available.

Step $k \geq 1$ Each agent "points to" his favorite available object. Each privately owned available object "points to" its owner and all other remaining objects point to the agent with the highest priority. Each agent in a cycle is assigned the object she points to and removed from the market with her assignment. If a private endowment is part of a cycle but its owner is not, then it becomes part of the social endowment in the next step. If there is at least one remaining agent, proceed with the next step.

An alternative representation of the generalized TTC

The you request my house - I get your turn (YRMH-IGYT) mechanism with an ordering of agents is defined as follows:

1. Let the agent with the top priority receive her best available object, the second agent his best available object among the remaining objects and so on, until someone requests a private endowment.
2. If the owner whose private endowment is requested has already received an object, then proceed the assignment to the next agent. Otherwise, insert her (the owner of the private endowment requestes) at the top of the priority order and proceed with the procedure.
3. A cycle, if any, is formed by the agents with a private endowment $i_1 \dots, i_k$ where i_1 points to the private endowment of agent i_2, who points to the private endowment of i_3, and so on. In such a case assign these objects by letting them exchange.

The properties of the generalized TTC

Theorem (Abdulkadiroglu and Sonmez, 1999)

Any member of the class of the generalized TTC mechanisms is individually rational, strategy-proof, and Pareto efficient.