A Short Course on Market Design and Matching

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Lecture 3: School Choice II

Tie-breaking

- Remember that there are priority classes. For each school, ties are broken by means of a lottery.
- Revisit the definition of stability:
  - A matching μ **violates the priority** of student $i \in I$ at school $s \in S$ if there exists another $j \in I$ such that $\mu(j) = s$, $s \; P_i \; \mu(i)$, and $i \succ_s j$.
  - A matching μ is **stable** if for each $i \in I$ and $s \in S$, it does not violate the priority of student i at school s.

Tie-breaking: An example

$\succ_{s_1}: i_1 - \{i_2, i_3\}$
$\succ_{s_2}: i_2 - \{i_1, i_3\}$
$\succ_{s_3}: i_3 - \{i_1, i_2\}$

$R_{i_1} : s_2 - s_1 - s_3$
$R_{i_2} : s_3 - s_2 - s_1$
$R_{i_3} : s_2 - s_3 - s_1$

- Suppose ties are broken favouring i_1 over i_2 over i_3.
- The SOSM for this tie-breaking is $\mu = \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$.
- But, it is Pareto dominated by the matching $\mu' = \{(i_1, s_1), (i_2, s_3), (i_3, s_2)\}$.
- Moreover, μ' is stable with respect to the the original (weak) priorities ($\succcurlyeq$) and students' preferences (R).

A constrained efficient solution

- Let μ be a stable matching for some fixed $\succcurlyeq$ and R.
- Student i *desires* school s if she prefers s to her assignment at μ, that is $s \; P_i \; \mu_i$.
- Let B_s denote the set of highest $\succcurlyeq_s$-priority students among those who desire s.
- A **stable improvement cycle (SIC)** consists of distinct students $i_1, \ldots, i_n = i_0$ such that
  1. each student in the cycle is assigned to a school,
  2. i_k desires $\mu(i_k + 1)$
  3. $i_k \in B_{\mu(i_k+1)}$

A constrained efficient solution

Theorem (Aytek and Erdil, 2008)

A stable matching μ is Pareto dominated by another stable matching ν if and only if μ admits a stable improvement cycle.

The SIC algorithm

Step 0 Select a strict priority structure $\succcurlyeq'$ by breaking the ties in $\succcurlyeq$. Run the DA algorithm and obtain a temporary matching μ_0.

Step k ≥ 1 **(k.1)** Given μ^{k-1}, let the G^k be a directed graph, where for each pair s and s' there is a directed edge from s to s' if and only if there is a student i who is matched to s under μ^{k-1}, and $i \in B_{s'}$.

(k.2) If there are any cycles in this directed graph, select one. For each edge from s to s' on this cycle, select a student $i \in B_{s'}$ with $\mu^{k-1}(i) = s$. Carry out this stable improvement cycle to obtain μ^k, and go to step (k+1).1. If there is no such cycle, then return μ^{k-1} as the outcome of the algorithm.

The SIC algorithm

- The SIC is a class of algorithms.
- Each member of the SIC class finds a SOSM.
- There is no strategy-proof mechanism which always finds a SOSM.

Tie-breaking: Single vs. multiple

- to break the ties separately at each school (multiple tie-breaking), or to break the ties the same way at every school (single tie-breaking) (single tiebreaking)
- which one is better? (Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match, Abdulkadiroglu, Pathak and Roth, 2009)
- a simulation study where tie-breakers are drawn independently from a uniform distribution a total of 250 times

Tie-breaking: Single vs. multiple

Theorem (Abdulkadiroglu et al., 2009)

Any matching that can be produced by deferred acceptance with some multiple tie-breaking, but that cannot be produced by deferred acceptance with any single tie-breaking, is not a student-optimal stable matching.

Tie-breaking: Single vs. multiple

Theorem (Abdulkadiroglu et al., 2009)

For any tie-breaking rule, there is no mechanism that is strategy-proof and Pareto dominates the DA outcome under this tie-breaking rule.

Tie-breaking simulations

TABLE 1—TIEBREAKING FOR GRADE 8 APPLICANTS IN NYC IN 2006–2007

Choice	Deferred acceptance single tiebreaking DA-STB (1)	Deferred acceptance multiple tiebreaking DA-MTB (2)	Student-optimal stable matching (3)	Improvement from DA-STB to student-optimal	Number of students (4)
1	32,105.3 (62.2)	29,849.9 (67.7)	32,701.5 (58.4)	+1	633.2 (32.1)
2	14,296.0 (53.2)	14,562.3 (59.0)	14,382.6 (50.9)	+2	338.6 (22.0)
3	9,279.4 (47.4)	9,859.7 (52.5)	9,208.6 (46.0)	+3	198.3 (15.5)
4	6,112.8 (43.5)	6,653.3 (47.5)	5,999.8 (41.4)	+4	125.6 (11.0)
5	3,988.2 (34.4)	4,386.8 (39.4)	3,883.4 (33.8)	+5	79.4 (8.9)
6	2,628.8 (29.6)	2,910.1 (33.5)	2,519.5 (28.4)	+6	51.7 (6.9)
7	1,732.7 (26.0)	1,919.1 (28.0)	1,654.6 (24.1)	+7	26.9 (5.1)
8	1,099.1 (23.3)	1,212.2 (26.8)	1,034.8 (22.1)	+8	17.0 (4.1)
9	761.9 (17.8)	817.1 (21.7)	716.7 (17.4)	+9	10.2 (3.1)
10	526.4 (15.4)	548.4 (19.4)	485.6 (15.1)	+10	4.7 (2.0)
11	348.0 (13.2)	353.2 (12.8)	316.3 (12.3)	+11	2.0 (1.1)
12	236.0 (10.9)	229.3 (10.5)	211.2 (10.4)		
Unassigned	5,613.4 (26.5)	5,426.7 (21.4)	5,613.4 (26.5)	Total:	1,487.5

Notes: Data from the main round of the New York City high school admissions process in 2006–2007 for students requesting an assignment for grade 9 (high school). Column 1 reports the average choice received distribution of applicants from the student-proposing deferred acceptance algorithm with single tiebreaking (DA-STB). Column 2 reports the average choice received distribution of applicants from the student-proposing deferred acceptance algorithm with school-specific tiebreaking. Column 3 reports the average choice received distribution of applicants in a student-optimal stable matching, which is computed from DA-STB followed by stable improvement cycles. Column 4 reports the average number of students and how many places on their rank order list students improve in the student-optimal stable matching relative to the matching produced by DA-STB. Columns 1, 2, 3, and 4 are based on 250 random draws. Simulation standard errors are reported in parentheses.

Tie-breaking simulations

- Significant costs of imposing strategy-proofness: about 1,500 students, or 1.9 percent of eighth graders, could be matched to schools they prefer over their assignment from deferred acceptance with single tie-breaking without harming any others, if the same preference information could be elicited by a nonstrategy-proof mechanism.
- The greater number of students obtaining one of their top choices in a similar simulation and in the first year of submitted preference data convinced New York City to employ a single tiebreaker in their assignment system.

Efficiency cost of stability

- A further (approximately) 4,300 students, or 5.5 percent of eighth graders in our sample, could improve their assignment if we relaxed the stability constraint, if the same rates of participation could be elicited by an unstable mechanism.

Creating diversity at schools

- The Jefferson County School District has an assignment plan that requires elementary schools to have between 15 and 50 percent of their students coming from a particular geographic area.
- In New York City, the Educational Option (EdOpt) schools have to accept students across different ability ranges. In particular, 16 percent of students that attend an EdOpt school must score above the grade level on the standardized English Language Arts test, 68 percent must score at the grade level, and the remaining 16 percent must score below the grade level.
- In practice, controlled school choice programs are often enforced by setting feasibility constraints with hard upper bounds and hard lower bounds for different student types.

Majority quotas

- Affirmative action with majority quotas (or simply majority quotas): prohibiting schools to admit more than a certain number ofmajority students.
- When there are two kinds of students (minority and majority), a quota for majority students may hurt minority students, the purported beneficiaries (Kojima, 2012). Why?

Majority quotas

- Consider a situation in which a school s is mostly desired by majorities. Then having a majority quota for s decreases the number of majority students who can be assigned to s even if there are empty seats. This, in turn, increases the competition for other schools and thus can even make the minority students worse off.

Minority reserves

- Schools assign minority reserves such that if the number of minority students in a school is less than its minority reserves, then any minority is preferred to any majority in that school.
- If there are not enough minority students to fill the reserves, majority students can still be admitted to fill up that school's reserved seats. Therefore, minority reserve mechanisms also avoid wasting the capacity in schools on top of resolving inefficiencies.
- The number of majority students can be more than its allotted share, which is the capacity of the school less the minority reserves, as long as there are no minority students who veto this match.

Minority reserves

- Affirmative action with minority reserves: schools give higher priority to minority students up to the point that the minorities fill the reserves. Minorities are on average better off with minority reserves while adverse effects on majorities are mitigated. (Ehlers et al, 2013)

Minority reserves

Step 1 Each student *i* applies to her first-choice school. Each school *s* first accepts as many as r_c^m minority applicants with the highest priorities if there are enough minority applicants. Then it accepts applicants with the highest priorities from the remaining applicants until its capacity is filled or the applicants are exhausted. The rest of the applicants, if any remain, are rejected by *s*.

Step k Each student *i* who got rejected at step $k-1$ applies to her next-choice school. Each school *s* considers the new applicants and students admitted tentatively at step $k-1$. Among these students, school *s* first accepts as many as r_c^m minority students with the highest priorities if there are enough minority students. Then it accepts students with the highest priorities from the remaining students. The remaining students, if any, are rejected by *s* . If there are no rejections, then stop.

Minority reserves

Theorem (Hafalir et al, 2013)

The student-proposing deferred acceptance algorithm with minority reserves produces a stable matching that assigns the best outcome among the set of stable matching outcomes for each student.

Minority reserves

Theorem (Hafalir et al, 2013)

Consider majority quotas q^M and minority reserves r^m such that $r^m = q - q^M$. Take a matching μ that is stable under majority quotas q^M. Then either μ is stable under minority reserves r^m or there exists a matching that is stable under minority reserves r^m that Pareto dominates μ.

Minority reserves

Theorem (Hafalir et al, 2013)

Consider minority reserves r^m. Let μ^r and μ be the matchings produced by the student-proposing deferred acceptance algorithm with or without minority reserves r^m, respectively, for a given preference profile. Then there exists at least one minority student i such that $\mu^r(i)\ R_i\ \mu(i)$.