A Short Course on Market Design and Matching

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Lecture 2: School Choice I

A simple theory

- a finite set of students $I = \{i_1, i_2, ..., i_n\}$;
- a finite set of schools $S = \{s_1, s_2, ..., s_m\}$;
- a strict priority profile of schools $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is the complete priority order of school s over I;
- a capacity vector $q = (q_s)_{s \in S}$ where q_s is the number of available seats at school s;
- a strict preference profile of students $P = (P_i)_{i \in I}$ such that P_i is student i's strict preferences over $S \cup \{\emptyset\}$, where $\emptyset$ stands for the option of being unassigned with $q_\emptyset = |I|$.

A simple theory

- A **matching** $\mu : I \to S \cup \{\emptyset\}$ is a function such that for each $s \in S$, $|\mu^{-1}(s)| \leq q_s$.
- A matching μ **violates the priority** of student $i \in I$ at school $s \in S$ if there exists another $j \in I$ such that $\mu(j) = s$, $s \; P_i \; \mu(i)$, and $i \succ_s j$.
- A matching μ is **stable** if for each $i \in I$ and $s \in S$, it does not violate the priority of student i at school s.

Do stable matchings exist?

Theorem (Gale and Shapley, 1962)

There always exists a stable matching.

- The proof relies on the convergence of an algorithm, the Deferred Acceptance (DA), and the fact that it always gives a *stable* matching.

The student-proposing DA Algorithm

Step 1 **(1.1)** Each student "applies" to her first choice school.
(1.2) Each school tentatively holds the most preferred applicants up to its quota and rejects all other students.

Step k $\geq$ 2 **(k.1)** Each student rejected in Step $k-1$ applies to her next highest choice.
(k.2) Each school considers both new applicants and the students (if any) held at Step $k-1$, tentatively holds the most preferred acceptable students up to its quota from the combined set of students, and rejects all other students.

The student-proposing DA Algorithm: An example

$\succ_{s_1}: i_1 - i_3 - i_2$
$\succ_{s_2}: i_2 - i_1 - i_3$
$\succ_{s_3}: i_2 - i_1 - i_3$

$R_{i_1} : s_2 - s_1 - s_3$
$R_{i_2} : s_1 - s_2 - s_3$
$R_{i_3} : s_1 - s_2 - s_3$

The SOSM

The matching given by the student-proposing DA is called as the Student Optimal Stable Matching (SOSM). Similarly, school-proposing DA results in the school-optimal stable matching.

Theorem (Gale and Shapley, 1962)

The SOSM Pareto dominates any other stable matching.

The SOSM

The last result clearly does not imply that the SOSM is Pareto efficient. Actually, the issue is deeper:

Theorem (Gale and Shapley, 1962)

Stability and Pareto efficiency are incompatible.

Strategic issues

- Preference revelation game
- In general, students can manipulate the system by revealing non-truthfully; that is, by misreporting, a student might be assigned to a better school than the one she would be assigned if she would have revealed her preferences truthfully.

Definition: strategy-proofness

A mechanism is **strategy-proof** if telling the true preferences is a dominant strategy.

Hospitals vs. schools

- What is the difference between hospital-intern matching and school-student matching?

Strategic issues

The student-proposing DA cannot be strategically-manipulated by the students.

Theorem (Roth, 1982)

The student-proposing DA is strategy-proof for students.

Strategic issues

How about the strategic behavior of the hospitals?

$\succ_{s_1}: i_2 - i_1$
$\succ_{s_2}: i_1 - i_2$

$R_{i_1}: s_1 - s_2$
$R_{i_2}: s_2 - s_1$

- Under true preferences, the student-proposing DA gives $\mu = \{(i_1, s_1), (i_2, s_2)\}$.
- If school s_1 reports that only student i_2 is acceptable, than $\mu' = \{(i_1, s_2), (i_2, s_1)\}$.
- Note that school s_1 "prefers" i_2 over i_1, thus better off by misreporting.

Strategic issues: one-to-one matching

Let each side of the market be strategic (e.g hospital-intern matching) and $q_s = 1$ for each s.

Theorem (Roth, 1982)

There is no stable mechanism that is strategy-proof

Strategic issues: many-to-one matching

Let each side of the market be strategic (e.g hospital-intern matching) and $q_s \geq 1$ for each s.

- No stable mechanism is strategy-proof for schools.
- Even school-proposing DA is not strategy-proof for schools.

Why do we care about strategic behaviour in real-life?

An example from the US:

- A widely used mechanism until 2005
- For each school a priority ordering is determined exogenously according to the following hierarchy:
  - First priority: sibling and walk zone
  - Second priority: sibling
  - Third priority: walk zone
  - Fourth priority: other students
  - A lottery number to break the ties
- Each student submits a preference ranking of the schools
- The student assignment is decided by the following mechanism (based on preferences and priorities):

The Boston Mechanism

Round 1 Only the first choices of the students are considered. For each school s, consider the students who have listed s as first choice and assign seats of school s to them one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her first choice.

Round k Consider the remaining students. Only the kth choices of these students are considered. For each school with still available seats, consider the students who have listed it as their kth choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her kth choice.

The Boston Mechanism: very manipulable!

- The Boston mechanism is not strategy-proof.
- *"Make a realistic, informed selection on the school you list as your first choice. It's the cleanest shot you will get at a school, but if you aim too high you might miss. Here's why: If the random computer selection rejects your first choice, your chances of getting your second choice school are greatly diminished. That's because you then fall in line behind everyone who wanted your second choice school as their first choice. You can fall even farther back in line as you get bumped down to your third, fourth and fifth choices."* (from an article published in St. Petersburgh Times)

The Boston Mechanism: very manipulable!

- Even if a student has very high priority at school s, she loses her priority to students who have top ranked school s unless she lists it as her top choice!
- The Boston mechanism gives parents strong incentives to over-rank schools where they have high priority.
- Are priorities respected under the Boston mechanism?

The DA replaces the Boston mechanism

- The Boston Globe published an article on Abdulkadiroğlu and Sönmez (2003), describing the flaws of the Boston mechanism, and advocating the adoption of the DA.
- Abdulkadiroğlu, Pathak, Roth and Sönmez presented to the BPS the case against the Boston mechanism, and proposed a strategy-proof alternative.
- The BPS staff was convinced that strategic behaviour was likely occurring, to the detriment of students and families.
- The BPS gave up the Boston mechanism and adopted the DA.