Quantum-geometric contribution to the Bogoliubov modes in a two-band Bose-Einstein condensate

M. Iskin
Department of Physics, Koç University, Rumelifeneri Yolu, 34450 Sarıyer, Istanbul, Turkey

(Received 28 October 2022; accepted 31 January 2023; published 17 February 2023)

We consider a weakly interacting Bose-Einstein condensate that is loaded into an optical lattice with a two-point basis and described by a two-band Bose-Hubbard model with generic one-body and two-body terms. By first projecting the system onto the lower Bloch band and then applying the Bogoliubov approximation to the resultant Hamiltonian, we show that the inverse effective-mass tensor of the superfluid carriers in the Bogoliubov spectrum has two physically distinct contributions. In addition to the usual inverse band-mass tensor that is originating from the intraband processes within the lower Bloch band, there is also a quantum-geometric contribution that is induced by the two-body interactions through the interband processes. We also discuss the conditions under which the latter contribution is expressed in terms of the quantum-metric tensor of the Bloch states, i.e., the natural Fubini-Study metric on the Bloch sphere.

DOI: 10.1103/PhysRevA.107.023313

I. INTRODUCTION

Recent theoretical efforts have firmly established that the quantum geometry of the underlying Bloch states lies at the heart of some multiband Fermi superfluids, e.g., Refs. [1–5] and the references therein. This is because, by controlling the effective-mass tensor of the superfluid carriers through the interband transitions [3,4,6], the quantum geometry can affect those superfluid properties that have explicit dependence on the carrier mass, e.g., superfluid weight and density, Berezinskii-Kosterlitz-Thouless transition temperature, and low-energy collective excitations [1–7]. While these predictions are based heavily on the multiband extension of the BCS formulation of superconductivity, their physical origins root deep into the two-body problem which is exactly tractable [8,9]. For instance, a pair of particles can still acquire a finite effective mass from the quantum geometry in the case when its unpaired constituents are completely localized and immobile, e.g., due to their diverging band mass in a flat band [8–10]. Thus the quantum-geometric interband mechanism resolves how superfluidity of Cooper pairs can prevail in a flat band [1].

Despite all this progress with Fermi superfluids, the defining effect of quantum geometry on multiband Bose superfluids is still in its infancy. For instance, in the case of a weakly interacting Bose-Einstein condensate (BEC) in a flat Bloch band, there are several multiband Bogoliubov analyses revealing that the quantum geometry dictates the speed of sound, quantum depletion, density-density correlations, and superfluid weight in fundamentally different ways [11–13]. In addition, there is a similar analysis for the spin-orbit-coupled Bose superfluids highlighting that the quantum geometry of the helicity states in such a two-band continuum model plays an analogous role [14], which is in accordance with the recent works on the spin-orbit-coupled Fermi superfluids [6,15].

Motivated by these recent works, and given that a crystal structure with a two-point basis is the minimal model to study quantum geometry of the Bloch states, here we consider a two-band Bose-Hubbard model with a generic single-particle spectrum and two-body interactions. By deriving the Bogoliubov spectrum for the low-energy quasiparticle excitations, we show that the interband processes that are induced by the two-body interactions give rise to a quantum-geometric contribution and dress the effective-mass tensor of the superfluid carriers. In the particular case when the BEC occurs uniformly within a unit cell, we also relate the geometric contribution to the quantum-metric tensor of the Bloch states. Similar to the Bogoliubov spectrum and superfluid weight and density, it is conceivable that all of the superfluid properties that depend on the carrier mass also have analogous quantum-geometric contributions. In fact, the energetic stability of a weakly interacting Bose superfluid relies solely on these contributions when the BEC occurs in a flat Bloch band [11–13].

The remaining text is organized as follows. In Sec. II we introduce the two-band Bose-Hubbard Hamiltonian in momentum space and project it to the lower Bloch band. Then in Sec. III we apply the Bogoliubov approximation to the projected Hamiltonian and extract the low-energy Bogoliubov modes. There we also relate the effective-mass tensor of the superfluid carriers to the quantum-metric tensor of the Bloch states and compare their relation with that of the Fermi superfluids. In Sec. IV we end the paper with a brief summary of our findings and an outlook. Benchmark with the extended Bose-Hubbard model is briefly discussed in Appendix A, and some example models with nontrivial geometry are presented in Appendix B.

II. BOSE-HUBBARD MODEL

In the presence of multiple sublattices, a generic Hamiltonian for the Bose-Hubbard model can be written as

$$\mathcal{H} = -\sum_{Si;S'i'} t_{Si;S'i'} c^\dagger_{Si} c_{S'i'} + \frac{1}{2}\sum_{Si} U_S c^\dagger_{Si} c^\dagger_{Si} c_{Si} c_{Si} + \sum_{\langle Si,S'i'\rangle} V_{Si,S'i'} c^\dagger_{Si} c^\dagger_{S'i'} c_{S'i'} c_{Si} - \mu \sum_{Si} c^\dagger_{Si} c_{Si}, \quad (1)$$

where the operator $c^\dagger_{Si}$ (c_{Si}) creates (annihilates) a particle at the sublattice site S in the unit cell i, the hopping parameter $t_{Si;S'i'}$ characterizes the tunneling between any pair of sites, the two-body terms U_S and $V_{Si,S'i'}$ describe, respectively, the on-site and off-site density-density interactions, and μ is the chemical potential. It is only the on-site interaction that is considered in the previous works [11–13]. The number of S sites determines the number of Bloch bands in the single-particle spectrum, and a lattice with a two-point basis (i.e., $S \in \{A, B\}$) is the minimal model to study the quantum geometry of underlying Bloch states. For the sake of clarity and for its simplicity, here we restrict our analysis to a two-band Bose-Hubbard model with generic single-particle spectrum and two-body interactions.

A. Two-band model in momentum space

To express the Bose-Hubbard Hamiltonian in momentum space, we Fourier expand the operators via $c^\dagger_{Si} = \frac{1}{\sqrt{N_c}}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}_{Si}} c^\dagger_{S\mathbf{k}}$, where N_c is the number of unit cells in the lattice, $\mathbf{k}$ is the crystal momentum in the first Brillouin zone, and $\mathbf{r}_{\mathbf{Si}}$ is the position of the site Si. Then a compact way to express the single-particle (hopping) term in general is

$$\mathcal{H}_0 = \sum_{\mathbf{k}} (c^\dagger_{A\mathbf{k}} \quad c^\dagger_{B\mathbf{k}}) \begin{pmatrix} d^0_{\mathbf{k}} + d^z_{\mathbf{k}} & d^x_{\mathbf{k}} - i d^y_{\mathbf{k}} \\ d^x_{\mathbf{k}} + i d^y_{\mathbf{k}} & d^0_{\mathbf{k}} - d^z_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{A\mathbf{k}} \\ c_{B\mathbf{k}} \end{pmatrix}, \quad (2)$$

where the field vector $\mathbf{d_k} = (d^x_{\mathbf{k}}, d^y_{\mathbf{k}}, d^z_{\mathbf{k}})$ that is coupled to a vector of Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ governs the sublattice degrees of freedom. Here $d^0_{\mathbf{k}}$, $d^x_{\mathbf{k}}$, $d^y_{\mathbf{k}}$, and $d^z_{\mathbf{k}}$ all depend on the specific details of the hopping parameters for a given lattice, and we do not make any assumption on their $\mathbf{k}$ dependence. Thus the Bloch bands are determined by the eigenvalues of the Hamiltonian matrix shown in Eq. (2), leading to $\varepsilon_{s\mathbf{k}} = d^0_{\mathbf{k}} + s d_{\mathbf{k}}$, where $s = \pm$ labels the upper and lower bands, and $d_{\mathbf{k}} = |\mathbf{d_k}|$ is the magnitude of the sublattice field. The corresponding Bloch states $|s\mathbf{k}\rangle$ can be represented as $|+,\mathbf{k}\rangle = (u_{\mathbf{k}},\ v_{\mathbf{k}} e^{i\varphi_{\mathbf{k}}})^{\mathrm{T}}$ and $|-,\mathbf{k}\rangle = (-v_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}},\ u_{\mathbf{k}})^{\mathrm{T}}$, where

$$u_{\mathbf{k}}/v_{\mathbf{k}} = \sqrt{\frac{1}{2} \pm \frac{d^z_{\mathbf{k}}}{2d_{\mathbf{k}}}}, \quad (3)$$

$$\varphi_{\mathbf{k}} = \arg\left(d^x_{\mathbf{k}} + i d^y_{\mathbf{k}}\right), \quad (4)$$

and T denotes a transpose.

Similarly a compact way to express the interaction terms in general is

$$\mathcal{H}_I = \frac{1}{2N_c}\sum_{SS'\mathbf{kk'q}} U_{SS'}(\mathbf{q}) c^\dagger_{S,\mathbf{k+q}} c^\dagger_{S',\mathbf{k'-q}} c_{S'\mathbf{k'}} c_{S\mathbf{k}}, \quad (5)$$

where $\mathbf{q}$ is the exchanged momentum between particles, and $U_{SS'}(\mathbf{q}) = U^*_{SS'}(-\mathbf{q}) = U_{S'S}(-\mathbf{q})$ by definition. Note that $U_{SS}(\mathbf{q})$ depends not only on the on-site interaction U_S but also on $V_{Si,Si'}$. Thus while the intra-sublattice interactions $U_{AA}(\mathbf{q})$ and $U_{BB}(\mathbf{q})$ are real and even functions of $\mathbf{q}$ in general, the inter-sublattice interaction $U_{AB}(\mathbf{q})$ can be complex. For instance, in the case of an extended Bose-Hubbard model with only nearest-neighbor hopping t, on-site repulsion U, and nearest-neighbor repulsion V, they can be written as $U_{AA}(\mathbf{q}) = U_{BB}(\mathbf{q}) = U$ and $U_{AB}(\mathbf{q}) = V\sum_{\delta_{nn}} e^{i\mathbf{q}\cdot\boldsymbol{\delta}_{nn}} = -(V/t)(d^x_{\mathbf{q}} - i d^y_{\mathbf{q}})$, where $\boldsymbol{\delta_{nn}}$ denotes the nearest neighbors of the A sublattice. Therefore, in this particular case, $d^x_{\mathbf{q}} = d^x_{-\mathbf{q}}$ is necessarily an even function of $\mathbf{q}$ while $d^y_{\mathbf{q}} = -d^y_{-\mathbf{q}}$ is an odd one. For example, $\boldsymbol{\delta_{nn}} \in \{(\pm a, 0), (0, \pm a)\}$ and $U_{AB}(\mathbf{q}) = 2V[\cos(q_x a) + \cos(q_y a)]$ in a square lattice, but $\boldsymbol{\delta_{nn}} \in \{(a, 0), (-a/2, \pm a\sqrt{3}/2)\}$ and $U_{AB}(\mathbf{q}) = V[\cos(q_x a) + 2\cos(q_x a/2)\cos(\sqrt{3}q_y a/2)] - iV[\sin(q_x a) - 2\sin(q_x a/2)\cos(\sqrt{3}q_y a/2)]$ in a honeycomb lattice.

The Bogoliubov spectrum for the total Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ can be obtained numerically through a straightforward application of the Bogoliubov approximation. Since the resultant Bogoliubov Hamiltonian matrix is 4×4, the spectrum has four branches; i.e., there are two quasiparticle and two quasihole bands that are related to each other through quasiparticle-quasihole symmetry. However, our main task here is to reveal a direct connection between the low-energy Bogoliubov modes and the quantum geometry of the Bloch states. Such a task can be achieved analytically by first projecting $\mathcal{H}$ onto the lower Bloch band and then applying the Bogoliubov approximation to the projected Hamiltonian.

B. Projection onto the lower Bloch band

Suppose BEC takes place at the Bloch state $|-,\mathbf{k_c}\rangle$. Next we assume that the energy gap $2d_{\mathbf{k_c}}$ between the lower and upper Bloch bands near this ground state is much larger than the interaction energy, and we project $\mathcal{H}$ to the lower band. Thus we first express $\mathcal{H}$ in the Bloch band basis, i.e., $c_{S\mathbf{k}} = \sum_s \langle S|s\mathbf{k}\rangle c_{s\mathbf{k}}$, and then we discard those terms that involve the upper band, i.e., we set $c_{S\mathbf{k}} \to \langle S|-,\mathbf{k}\rangle c_{-,\mathbf{k}}$ or more explicitly $c_{A\mathbf{k}} \to -v_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} c_{-,\mathbf{k}}$ and $c_{B\mathbf{k}} \to u_{\mathbf{k}} c_{-,\mathbf{k}}$. Here $c_{s\mathbf{k}}$ annihilates a particle from the Bloch state $|s\mathbf{k}\rangle$. This procedure leads to the projected Hamiltonian

$$\mathcal{H}_P = \sum_{\mathbf{k}} (\varepsilon_{-,\mathbf{k}} - \mu) c^\dagger_{-,\mathbf{k}} c_{-,\mathbf{k}} + \frac{1}{2N_c}\sum_{\mathbf{kk'q}} f^{\mathbf{k'},\mathbf{k}}_{\mathbf{k+q},\mathbf{k'-q}} c^\dagger_{-,\mathbf{k+q}} c^\dagger_{-,\mathbf{k'-q}} c_{-,\mathbf{k'}} c_{-,\mathbf{k}}, \quad (6)$$

where the second term describes the dressed density-density interactions in the lower Bloch band with the effective interaction amplitude $f^{\mathbf{k'},\mathbf{k}}_{\mathbf{k+q},\mathbf{k'-q}} = \sum_{SS'} U_{SS'}(\mathbf{q}) \langle -,\mathbf{k+q}|S\rangle\langle -,\mathbf{k'}-\mathbf{q}|S'\rangle\langle S'|-,\mathbf{k'}\rangle\langle S|-,\mathbf{k}\rangle$. In terms of the Bloch factors, this effective interaction becomes

$$f_{\mathbf{k+q},\mathbf{k'-q}}^{\mathbf{k'},\mathbf{k}} = U_{AA}(\mathbf{q})v_{\mathbf{k+q}}v_{\mathbf{k'-q}}v_{\mathbf{k'}}v_{\mathbf{k}}e^{i(\varphi_{\mathbf{k+q}}+\varphi_{\mathbf{k'-q}}-\varphi_{\mathbf{k'}}-\varphi_{\mathbf{k}})} + U_{BB}(\mathbf{q})u_{\mathbf{k+q}}u_{\mathbf{k'-q}}u_{\mathbf{k'}}u_{\mathbf{k}} + U_{AB}(\mathbf{q})v_{\mathbf{k+q}}u_{\mathbf{k'-q}}u_{\mathbf{k'}}v_{\mathbf{k}}e^{i(\varphi_{\mathbf{k'+q}}-\varphi_{\mathbf{k}})} + U_{BA}(\mathbf{q})u_{\mathbf{k+q}}v_{\mathbf{k'-q}}v_{\mathbf{k'}}u_{\mathbf{k}}e^{i(\varphi_{\mathbf{k'-q}}-\varphi_{\mathbf{k'}})}. \quad (7)$$

Note that the $\mathbf{q}$ dependence of $f_{\mathbf{k+q},\mathbf{k'-q}}^{\mathbf{k'},\mathbf{k}}$ remains there even in the absence of the off-site interaction term $V_{Si,S'i'}$ in Eq. (1).

Equation (6) is expected to be quantitatively accurate in describing the low-energy physics when the occupation of the upper band is negligible, e.g., in the weakly interacting limit. For instance, in the case of spin-orbit-coupled Bose gases, the corresponding $\mathcal{H}_P$ works surprisingly well as it perfectly reproduces the Bogoliubov spectrum not only at low momenta but also near the band touchings [14]. See also Appendix A.

III. BOGOLIUBOV THEORY

In the Bogoliubov approximation, we first replace the creation and annihilation operators in accordance with $c_{-,\mathbf{k}} = \sqrt{N_0}\delta_{\mathbf{k}\mathbf{k}_c} + \tilde{c}_{-,\mathbf{k}}$, where N_0 is the number of condensed particles at the Bloch state $|-,\mathbf{k}_c\rangle$, $\delta_{\mathbf{k}\mathbf{k}_c}$ is a Kronecker delta, and the operator $\tilde{c}_{-,\mathbf{k}}$ denotes the fluctuations on top of the many-body ground state. Then we set the first-order fluctuations to zero and discard the third- and fourth-order fluctuations. The former condition gives $\mu = \varepsilon_{-,\mathbf{k}_c} + n_0 f_{\mathbf{k}_c\mathbf{k}_c}^{\mathbf{k}_c\mathbf{k}_c}$, leading to

$$\mu = \varepsilon_{-,\mathbf{k}_c} + n_0\big[U_{AA}(\mathbf{0})v_{\mathbf{k}_c}^4 + U_{BB}(\mathbf{0})u_{\mathbf{k}_c}^4 + 2U_{AB}(\mathbf{0})u_{\mathbf{k}_c}^2 v_{\mathbf{k}_c}^2\big], \quad (8)$$

where $U_{AB}(\mathbf{0}) = U_{BA}(\mathbf{0})$ is real by definition and $n_0 = N_0/N_c$ is the condensate filling per unit cell. Note that the condensate filling within a unit cell (i.e., on sublattices A and B) is not necessarily uniform unless $u_{\mathbf{k}_c} = v_{\mathbf{k}_c} = 1/\sqrt{2}$ (i.e., $d_{\mathbf{k}_c}^z = 0$) is favored by the interactions.

A. Bogoliubov Hamiltonian

The second-order fluctuations are described by the Bogoliubov Hamiltonian

$$\mathcal{H}_\mathrm{B} = \frac{1}{2}\sum_{\mathbf{q}}{}' (\tilde{c}_{-,\mathbf{k}_c+\mathbf{q}}^\dagger \quad \tilde{c}_{-,\mathbf{k}_c-\mathbf{q}})\begin{pmatrix} h_\mathbf{q}^\mathrm{pp} & h_\mathbf{q}^\mathrm{ph} \\ h_\mathbf{q}^\mathrm{hp} & h_\mathbf{q}^\mathrm{hh}\end{pmatrix}\begin{pmatrix}\tilde{c}_{-,\mathbf{k}_c+\mathbf{q}} \\ \tilde{c}_{-,\mathbf{k}_c-\mathbf{q}}^\dagger\end{pmatrix}, \quad (9)$$

where the diagonal elements $h_\mathbf{q}^\mathrm{hh} = h_{-\mathbf{q}}^\mathrm{pp}$ are given by $h_\mathbf{q}^\mathrm{pp} = \varepsilon_{-,\mathbf{k}_c+\mathbf{q}} - \mu + \frac{n_0}{2}(f_{\mathbf{k}_c,\mathbf{k}_c+\mathbf{q}}^{\mathbf{k}_c,\mathbf{k}_c+\mathbf{q}} + f_{\mathbf{k}_c+\mathbf{q},\mathbf{k}_c}^{\mathbf{k}_c+\mathbf{q},\mathbf{k}_c} + f_{\mathbf{k}_c+\mathbf{q},\mathbf{k}_c}^{\mathbf{k}_c,\mathbf{k}_c+\mathbf{q}} + f_{\mathbf{k}_c,\mathbf{k}_c+\mathbf{q}}^{\mathbf{k}_c+\mathbf{q},\mathbf{k}_c})$, and the off-diagonal elements $h_\mathbf{q}^\mathrm{hp} = (h_\mathbf{q}^\mathrm{ph})^*$ are given by $h_\mathbf{q}^\mathrm{ph} = \frac{n_0}{2}(f_{\mathbf{k}_c+\mathbf{q},\mathbf{k}_c-\mathbf{q}}^{\mathbf{k}_c,\mathbf{k}_c} + f_{\mathbf{k}_c-\mathbf{q},\mathbf{k}_c+\mathbf{q}}^{\mathbf{k}_c,\mathbf{k}_c})$. The prime symbol in Eq. (9) indicates that the summation excludes the condensed state. In terms of the Bloch factors, the matrix elements become

$$h_\mathbf{q}^\mathrm{pp} = \varepsilon_{-,\mathbf{k}_c+\mathbf{q}} - \mu + n_0\big\{[U_{AA}(\mathbf{q}) + U_{AA}(\mathbf{0})]v_{\mathbf{k}_c}^2 v_{\mathbf{k}_c+\mathbf{q}}^2 + [U_{BB}(\mathbf{q}) + U_{BB}(\mathbf{0})]u_{\mathbf{k}_c}^2 u_{\mathbf{k}_c+\mathbf{q}}^2 + U_{AB}(\mathbf{0})\big(u_{\mathbf{k}_c}^2 v_{\mathbf{k}_c+\mathbf{q}}^2 + v_{\mathbf{k}_c}^2 u_{\mathbf{k}_c+\mathbf{q}}^2\big) + 2\mathrm{Re}[U_{AB}(\mathbf{q})e^{i(\varphi_{\mathbf{k}_c+\mathbf{q}}-\varphi_{\mathbf{k}_c})}]u_{\mathbf{k}_c}v_{\mathbf{k}_c}u_{\mathbf{k}_c+\mathbf{q}}v_{\mathbf{k}_c+\mathbf{q}}\big\}, \quad (10)$$

$$h_\mathbf{q}^\mathrm{ph} = n_0\big[U_{AA}(\mathbf{q})e^{i(\varphi_{\mathbf{k}_c+\mathbf{q}}+\varphi_{\mathbf{k}_c-\mathbf{q}}-2\varphi_{\mathbf{k}_c})}v_{\mathbf{k}_c}^2 v_{\mathbf{k}_c+\mathbf{q}}v_{\mathbf{k}_c-\mathbf{q}} + U_{BB}(\mathbf{q})u_{\mathbf{k}_c}^2 u_{\mathbf{k}_c+\mathbf{q}}u_{\mathbf{k}_c-\mathbf{q}} + U_{AB}(\mathbf{q})e^{i(\varphi_{\mathbf{k}_c+\mathbf{q}}-\varphi_{\mathbf{k}_c})}u_{\mathbf{k}_c}v_{\mathbf{k}_c}u_{\mathbf{k}_c-\mathbf{q}}v_{\mathbf{k}_c+\mathbf{q}} + U_{AB}(-\mathbf{q})e^{i(\varphi_{\mathbf{k}_c-\mathbf{q}}-\varphi_{\mathbf{k}_c})}u_{\mathbf{k}_c}v_{\mathbf{k}_c}u_{\mathbf{k}_c+\mathbf{q}}v_{\mathbf{k}_c-\mathbf{q}}\big], \quad (11)$$

for the particle-particle and particle-hole sectors, where Re denotes the real part.

The Bogoliubov spectrum $E_{s\mathbf{q}}$ for Eq. (9) is determined by the eigenvalues of $\tau_z\mathbf{h}_\mathbf{q}$ so that the bosonic commutation rules are properly taken into account, where τ_z is the third Pauli matrix in the particle-hole space and $\mathbf{h}_\mathbf{q}$ is the Hamiltonian matrix shown in Eq. (9). Thus the spectrum has two modes for a given $\mathbf{q}$, i.e.,

$$E_{s\mathbf{q}} = A_\mathbf{q} + s\sqrt{B_\mathbf{q}^2 - \left|h_\mathbf{q}^\mathrm{ph}\right|^2}, \quad (12)$$

$$A_\mathbf{q}/B_\mathbf{q} = \frac{h_\mathbf{q}^\mathrm{pp} \mp h_\mathbf{q}^\mathrm{hh}}{2}, \quad (13)$$

where $s = \pm$ denotes, respectively, the quasiparticle and quasihole branches in the first line. Here $A_\mathbf{q}$ is odd in $\mathbf{q}$, and $B_\mathbf{q}$ and $h_\mathbf{q}^\mathrm{ph}$ are even in $\mathbf{q}$, so that the quasiparticle-quasihole symmetry $E_{+,\mathbf{q}} = -E_{-,-\mathbf{q}}$ manifests in the spectrum.

B. Low-energy Bogoliubov excitations

Since our primary objective is to derive an analytical expression for the low-energy Bogoliubov modes, next we calculate $E_{s\mathbf{q}}$ accurately up to first order in $\mathbf{q}$. For this purpose we first recall that $U_{AB}(\mathbf{q}) = U_{AB}^*(-\mathbf{q})$, and therefore $U_{AB}(\mathbf{0})$ is always real. Furthermore, given that the zeroth-order contribution from the imaginary part $\mathrm{Im}[h_\mathbf{0}^\mathrm{ph}] = 0$ vanishes, its second-order contribution (which contributes to the square-root term in $E_{s\mathbf{q}}$ at quartic order in $\mathbf{q}$) is not needed for the determination of the effective-mass tensor of the superfluid carriers. Thus we may simply substitute $|h_\mathbf{q}^\mathrm{ph}|^2 \to C_\mathbf{q}^2$ for the low-$\mathbf{q}$ modes, where

$$C_\mathbf{q} = n_0\big\{v_{\mathbf{k}_c}^2 v_{\mathbf{k}_c+\mathbf{q}}v_{\mathbf{k}_c-\mathbf{q}}U_{AA}(\mathbf{q}) + u_{\mathbf{k}_c}^2 u_{\mathbf{k}_c+\mathbf{q}}u_{k_c-q}U_{BB}(\mathbf{q}) + u_{\mathbf{k}_c}v_{\mathbf{k}_c}u_{\mathbf{k}_c-\mathbf{q}}v_{\mathbf{k}_c+\mathbf{q}}\mathrm{Re}[U_{AB}(\mathbf{q})e^{i(\varphi_{\mathbf{k}_c+\mathbf{q}}-\varphi_{\mathbf{k}_c})}] + u_{\mathbf{k}_c}v_{\mathbf{k}_c}u_{\mathbf{k}_c+\mathbf{q}}v_{\mathbf{k}_c-\mathbf{q}}\mathrm{Re}[U_{AB}(-\mathbf{q})e^{i(\varphi_{\mathbf{k}_c-\mathbf{q}}-\varphi_{\mathbf{k}_c})}]\big\} \quad (14)$$

is taken as real up to second-order in $\mathbf{q}$. Note here that, since $\varphi_{\mathbf{k}_c+\mathbf{q}} + \varphi_{\mathbf{k}_c-\mathbf{q}} - 2\varphi_{\mathbf{k}_c}$ is even in $\mathbf{q}$ and vanishes at the zeroth order, the $\mathbf{q}$ dependence coming from its cosine is at least quartic order, and therefore it is dropped from the first term as well. Then we only need the expansion of $A_\mathbf{q} = \sum_\ell A_\ell q_\ell + O(\mathbf{q}^3)$, up to first order in $\mathbf{q}$, and the expansions of $B_\mathbf{q} = B_\mathbf{0} + (1/2)\sum_{\ell\ell'} B_{\ell\ell'}q_\ell q_{\ell'} + O(\mathbf{q}^4)$ and $C_\mathbf{q} = C_\mathbf{0} + (1/2)\sum_{\ell\ell'} C_{\ell\ell'}q_\ell q_{\ell'} + O(\mathbf{q}^4)$, up to second orders in $\mathbf{q}$. Here q_ℓ refers to the ℓth component of the $\mathbf{q} = (q_x, q_y, q_z)$ vector,

$A_\ell = (\partial A_{\mathbf{q}}/\partial q_\ell)_{\mathbf{q}=\mathbf{0}}$, $B_{\ell\ell'} = (\partial^2 B_{\mathbf{q}}/\partial q_\ell \partial q_{\ell'})_{\mathbf{q}=\mathbf{0}}$, and similarly $C_{\ell\ell'} = (\partial^2 C_{\mathbf{q}}/\partial q_\ell \partial q_{\ell'})_{\mathbf{q}=\mathbf{0}}$.

The zeroth-order coefficients $B_{\mathbf{0}} = \varepsilon_{\mathbf{k}_c} - \mu + 2n_0[v_{\mathbf{k}_c}^4 U_{AA}(\mathbf{0}) + u_{\mathbf{k}_c}^4 U_{BB}(\mathbf{0}) + u_{\mathbf{k}_c}^2 v_{\mathbf{k}_c}^2 U_{AB}(\mathbf{0})]$ and $C_{\mathbf{0}} = n_0[v_{\mathbf{k}_c}^4 U_{AA}(\mathbf{0}) + u_{\mathbf{k}_c}^4 U_{BB}(\mathbf{0}) + 2u_{\mathbf{k}_c}^2 v_{\mathbf{k}_c}^2 U_{AB}(\mathbf{0})]$ are equal to each other due to Eq. (8), which guarantees that the $\mathbf{q} = \mathbf{0}$ mode is gapless, i.e., $E_{s\mathbf{0}} = 0$. Thus we find

$$E_{s\mathbf{q}} = \sum_\ell A_\ell q_\ell + s\sqrt{B_{\mathbf{0}} \sum_{\ell\ell'} (B_{\ell\ell'} - C_{\ell\ell'}) q_\ell q_{\ell'}} + O(\mathbf{q}^2) \quad (15)$$

for the low-energy quasiparticle and quasihole excitations, where

$$A_\ell = \dot{\varepsilon}_{\mathbf{k}_c}^\ell + n_0 \big[4U_{AA}(\mathbf{0}) v_{\mathbf{k}_c}^3 \dot{v}_{\mathbf{k}_c}^\ell + 4U_{BB}(\mathbf{0}) u_{\mathbf{k}_c}^3 \dot{u}_{\mathbf{k}_c}^\ell + \dot{U}_{AA}^\ell(\mathbf{0}) v_{\mathbf{k}_c}^4 + \dot{U}_{BB}^\ell(\mathbf{0}) u_{\mathbf{k}_c}^4 + 2\dot{U}_{AB}^\ell(\mathbf{0}) u_{\mathbf{k}_c}^2 v_{\mathbf{k}_c}^2 + 4U_{AB}(\mathbf{0}) u_{\mathbf{k}_c} v_{\mathbf{k}_c} \big(u_{\mathbf{k}_c} \dot{v}_{\mathbf{k}_c}^\ell + v_{\mathbf{k}_c} \dot{u}_{\mathbf{k}_c}^\ell\big)\big] \quad (16)$$

is the coefficient of the linear term, and

$$B_{\ell\ell'} - C_{\ell\ell'} = (M^{-1})_{\ell\ell'} + n_0 \big\{2U_{AA}(\mathbf{0}) v_{\mathbf{k}_c}^2 \big(v_{\mathbf{k}_c} \ddot{v}_{\mathbf{k}_c}^{\ell\ell'} + 3\dot{v}_{\mathbf{k}_c}^\ell \dot{v}_{\mathbf{k}_c}^{\ell'}\big) + 2U_{BB}(\mathbf{0}) u_{\mathbf{k}_c}^2 \big(u_{\mathbf{k}_c} \ddot{u}_{\mathbf{k}_c}^{\ell\ell'} + 3\dot{u}_{\mathbf{k}_c}^\ell \dot{u}_{\mathbf{k}_c}^{\ell'}\big) + 2\big[\dot{U}_{AA}^\ell(\mathbf{0}) \dot{v}_{\mathbf{k}_c}^{\ell'} + \dot{U}_{AA}^{\ell'}(\mathbf{0}) \dot{v}_{\mathbf{k}_c}^\ell\big] v_{\mathbf{k}_c}^3 + 2\big[\dot{U}_{BB}^\ell(\mathbf{0}) \dot{u}_{\mathbf{k}_c}^{\ell'} + \dot{U}_{BB}^{\ell'}(\mathbf{0}) \dot{u}_{\mathbf{k}_c}^\ell\big] u_{\mathbf{k}_c}^3 + 2U_{AB}(\mathbf{0}) \big[u_{\mathbf{k}_c}^2 \big(v_{\mathbf{k}_c} \ddot{v}_{\mathbf{k}_c}^{\ell\ell'} + \dot{v}_{\mathbf{k}_c}^\ell \dot{v}_{\mathbf{k}_c}^{\ell'}\big) + v_{\mathbf{k}_c}^2 \big(u_{\mathbf{k}_c} \ddot{u}_{\mathbf{k}_c}^{\ell\ell'} + \dot{u}_{\mathbf{k}_c}^\ell \dot{u}_{\mathbf{k}_c}^{\ell'}\big) + 2u_{\mathbf{k}_c} v_{\mathbf{k}_c} \big(\dot{u}_{\mathbf{k}_c}^\ell \dot{v}_{\mathbf{k}_c}^{\ell'} + \dot{v}_{\mathbf{k}_c}^\ell \dot{u}_{\mathbf{k}_c}^{\ell'}\big)\big] + 4\mathrm{Re}\big[\dot{U}_{AB}^\ell(\mathbf{0}) \dot{u}_{\mathbf{k}_c}^{\ell'} + \dot{U}_{AB}^{\ell'}(\mathbf{0}) \dot{u}_{\mathbf{k}_c}^\ell\big] u_{\mathbf{k}_c} v_{\mathbf{k}_c}^2 \big\} \quad (17)$$

is the coefficient of the quadratic term inside the square root. Here $(M^{-1})_{\ell\ell'} = (\partial^2 \varepsilon_{-,\mathbf{k}_c+\mathbf{q}}/\partial q_\ell \partial q_{\ell'})_{\mathbf{q}=\mathbf{0}}$ is the matrix element of the inverse band-mass tensor $\mathbf{M}^{-1}$ for a particle in the lower Bloch band, $\dot{u}_{\mathbf{k}_c}^\ell = (\partial u_{\mathbf{k}_c+\mathbf{q}}/\partial q_\ell)_{\mathbf{q}=\mathbf{0}}$, $\ddot{u}_{\mathbf{k}_c}^{\ell\ell'} = (\partial^2 u_{\mathbf{k}_c+\mathbf{q}}/\partial q_\ell \partial q_{\ell'})_{\mathbf{q}=\mathbf{0}}$, and $\dot{U}_{SS'}^\ell(\mathbf{0}) = [\partial U_{SS'}(\mathbf{q})/\partial q_\ell]_{\mathbf{q}=\mathbf{0}}$. Note that Eq. (17) can be interpreted as the inverse effective-mass tensor for the superfluid carriers dressed by the presence of an upper Bloch band [14].

Equations (16) and (17) can be simplified considerably as follows. Since $U_{AB}(\mathbf{q}) = U_{AB}^*(-\mathbf{q})$, we first note that $\mathrm{Re}[U_{AB}(\mathbf{q})] = \mathrm{Re}[U_{AB}^*(-\mathbf{q})]$ is an even function of $\mathbf{q}$, and therefore we take $\mathrm{Re}[\dot{U}_{AB}^\ell(\mathbf{0})] = 0$. Similarly $U_{SS}(\mathbf{q})$ is also an even function of $\mathbf{q}$, and therefore we take $\dot{U}_{SS}^\ell(\mathbf{0}) = 0$. Furthermore we suppose $U_{AA}(\mathbf{0}) = U_{BB}(\mathbf{0}) = \mathcal{U}$ are equal for both sublattices, $U_{AB}(\mathbf{0}) = \mathcal{V}$, and $\dot{\varepsilon}_{\mathbf{k}_c}^\ell = 0$. It is also convenient to substitute $u_{\mathbf{q}} = \cos(\theta_{\mathbf{q}}/2)$ and $v_{\mathbf{q}} = \sin(\theta_{\mathbf{q}}/2)$ without loss of generality as they satisfy $u_{\mathbf{q}}^2 + v_{\mathbf{q}}^2 = 1$. Note that $\theta_{\mathbf{q}}$ and $\varphi_{\mathbf{q}}$ correspond, respectively, to the azimuthal and polar angles on the Bloch sphere. With these simplifications, we find that

$$A_\ell = -\frac{n_0}{2} \mathcal{U} \sin(2\theta_{\mathbf{k}_c}) \dot{\theta}_{\mathbf{k}_c}^\ell, \quad (18)$$

$$B_{\mathbf{0}} = n_0 \mathcal{U} + \frac{n_0}{2} (\mathcal{V} - \mathcal{U}) \sin^2 \theta_{\mathbf{k}_c}, \quad (19)$$

$$B_{\ell\ell'} - C_{\ell\ell'} = (M^{-1})_{\ell\ell'} + \frac{n_0}{4} (\mathcal{V} - \mathcal{U}) \times \big[\sin(2\theta_{\mathbf{k}_c}) \ddot{\theta}_{\mathbf{k}_c}^{\ell\ell'} + 2\cos(2\theta_{\mathbf{k}_c}) \dot{\theta}_{\mathbf{k}_c}^\ell \dot{\theta}_{\mathbf{k}_c}^{\ell'}\big] \quad (20)$$

are the desired expansion coefficients in general, where $\dot{\theta}_{\mathbf{k}_c}^\ell = (\partial \theta_{\mathbf{k}_c+\mathbf{q}}/\partial q_\ell)_{\mathbf{q}=\mathbf{0}}$ and $\ddot{\theta}_{\mathbf{k}_c}^{\ell\ell'} = (\partial^2 \theta_{\mathbf{k}_c+\mathbf{q}}/\partial q_\ell \partial q_{\ell'})_{\mathbf{q}=\mathbf{0}}$. Equation (20) reveals that the dressing of the effective-mass tensor is caused by the presence of a second band in the Bloch spectrum and that it has a peculiar dependence on the geometry of the Bloch sphere. Note that Eqs. (18), (19), and (20) do not depend on $\varphi_{\mathbf{q}}$. See also a related discussion at the end of Sec. III in Ref. [13].

It is important to emphasize that these generic expressions are valid and applicable to a broad range of two-band Bose-Hubbard models. In the particular case when $\theta_{\mathbf{k}_c} = \pi/2$, i.e., when $d_{\mathbf{k}_c}^z = 0$, Eq. (15) can be written as

$$E_{s\mathbf{q}} = s\sqrt{n_0 \frac{\mathcal{U} + \mathcal{V}}{2} \sum_{\ell\ell'} \big(M_{\text{eff}}^{-1}\big)_{\ell\ell'} q_\ell q_{\ell'}} + O(\mathbf{q}^2), \quad (21)$$

$$\big(M_{\text{eff}}^{-1}\big)_{\ell\ell'} = (M^{-1})_{\ell\ell'} + n_0 \frac{\mathcal{U} - \mathcal{V}}{2} \dot{\theta}_{\mathbf{k}_c}^\ell \dot{\theta}_{\mathbf{k}_c}^{\ell'}, \quad (22)$$

where $(M_{\text{eff}}^{-1})_{\ell\ell'}$ is the matrix element of the inverse effective-mass tensor $\mathbf{M}_{\text{eff}}^{-1}$ for the superfluid carriers. This case corresponds to a uniform condensate filling on sublattices A and B since $u_{\mathbf{k}_c} = v_{\mathbf{k}_c} = 1/\sqrt{2}$. Given that both sublattices are equally populated, $n_0/2$ corresponds to the condensate filling per lattice site in the system and hence to the proper prefactor for the effective mass. For instance, when $\mathcal{U} > \mathcal{V}$, the ground state of a flat-band BEC is expected to be uniform over the unit cell as this configuration minimizes the repulsive interactions [16]. We note in passing that, since $\dot{\theta}_{\mathbf{q}}^\ell$ is trivially zero when $d_{\mathbf{q}}^z = 0$ for every $\mathbf{q}$ in the entire Brillouin zone, the presence of a finite geometric contribution relies on a nontrivial $d_{\mathbf{q}}^z$ to begin with. For instance, in the case of bipartite lattices, next-nearest-neighbor hopping processes may give rise to such an intra-sublattice term in the Bloch Hamiltonian. See Appendix B for example models. Unless $d_{\mathbf{q}}^z$ is coupled with a $d_{\mathbf{q}}^x$ and/or $d_{\mathbf{q}}^y$ term in the Bloch Hamiltonian, $\theta_{\mathbf{q}} = \{0, \pi\}$ for every $\mathbf{q}$ in the entire Brillouin zone, and therefore, $\dot{\theta}_{\mathbf{k}_c}^\ell = 0$ becomes trivial. Furthermore the geometric dressing is also trivial when $\mathcal{U} = \mathcal{V}$, whose physical significance is not obvious.

On the other hand when $\theta_{\mathbf{k}_c} = \{0, \pi\}$, i.e., when $d_{\mathbf{k}_c}^x = 0 = d_{\mathbf{k}_c}^y$ and $d_{\mathbf{k}_c}^z \gtrless 0$, Eq. (15) reduces to $E_{s\mathbf{q}} = s\sqrt{n_0 \mathcal{U} \sum_{\ell\ell'} [(M^{-1})_{\ell\ell'} + n_0 \frac{\mathcal{V} - \mathcal{U}}{2} \dot{\theta}_{\mathbf{k}_c}^\ell \dot{\theta}_{\mathbf{k}_c}^{\ell'}] q_\ell q_{\ell'}} + O(\mathbf{q}^2)$. While the $\theta_{\mathbf{k}_c} = 0$ case with $u_{\mathbf{k}_c} = 1$ and $v_{\mathbf{k}_c} = 0$ corresponds to a condensate filling that is entirely on sublattice B, $\theta_{\mathbf{k}_c} = \pi$ case with $u_{\mathbf{k}_c} = 0$ and $v_{\mathbf{k}_c} = 1$ corresponds to a condensate filling that is entirely on sublattice A. Given that one of the sublattices is empty, n_0 corresponds to the condensate filling per site for the occupied sublattice, which explains the difference between the prefactor of the effective mass here and in

Eq. (21). Thus, since the condensate filling has the structure of a charge-density-wave pattern in both of these extreme cases, the inter-sublattice interaction $\mathcal{V}$ must disappear from the Bogoliubov spectrum because one of the sublattices is not macroscopically occupied. In fact, it can be shown that $\dot{\theta}^\ell_{\mathbf{k_c}} = 2\dot{v}^\ell_{\mathbf{k_c}}/u_{\mathbf{k_c}} = 0$ in general for both of these extreme cases, which makes their geometric dressing trivial. For instance, in the case of the extended Bose-Hubbard model that is discussed in Appendix A, such a density-wave superfluid (i.e., a supersolid) may occur when the nearest-neighbor repulsion $\mathcal{V}$ is sufficiently larger than the on-site one on a bipartite lattice, and the asymmetric occupation of the sublattices can become as dramatic only in the $\mathcal{V} \gg \mathcal{U}$ limit [17,18].

We emphasize that the bare band-mass tensor in Eq. (22) and its dressing terms have completely different physical origins. While the usual term $(M^{-1})_{\ell\ell'}$ is associated with the intraband processes within the lower Bloch band in which the BEC occurs, the dressing terms are related to the interband processes that are induced by the interactions. That is why they have an overall factor of $\mathcal{U}$ and/or $\mathcal{V}$ in the front. The interband terms are quite peculiar because they depend not only on the Bloch bands but also on the Bloch states themselves, i.e., on the geometry of the Bloch sphere. For this reason they are claimed to have a quantum-geometric origin [11–14]. Their critical roles in Eqs. (20) and (21) are to renormalize and dress the inverse effective-mass tensor of the superfluid carriers. Next we show that the geometric contribution of Eq. (22) can be related to the quantum-metric tensor of the underlying Bloch states under some specific conditions.

C. Connection to the quantum-metric tensor

The quantum-metric tensor corresponds to the real part of the quantum-geometric tensor [19]. For instance, in the case of a multiband Bloch Hamiltonian, it can be expressed in general as $g^{s\mathbf{k}}_{\ell\ell'} = \mathrm{Re}[(\partial\langle s\mathbf{k}|/\partial k_\ell)(\mathbb{I} - |s\mathbf{k}\rangle\langle s\mathbf{k}|)(\partial|s\mathbf{k}\rangle/\partial k_{\ell'})]$, where $|s\mathbf{k}\rangle$ corresponds to the Bloch state for band s at momentum $\mathbf{k}$, and $\mathbb{I} = \sum_s |s\mathbf{k}\rangle\langle s\mathbf{k}|$ denotes the identity operator for a given $\mathbf{k}$. In the case of two-band lattices where $s = \pm$, it can be shown that $g^{+,\mathbf{k}}_{\ell\ell'} = g^{-,\mathbf{k}}_{\ell\ell'} = g^{\mathbf{k}}_{\ell\ell'}$ is given by

$$g^{\mathbf{k}}_{\ell\ell'} = \frac{1}{4}\dot{\theta}^\ell_{\mathbf{k}}\dot{\theta}^{\ell'}_{\mathbf{k}} + \frac{\sin^2\theta_{\mathbf{k}}}{4}\dot{\varphi}^\ell_{\mathbf{k}}\dot{\varphi}^{\ell'}_{\mathbf{k}}, \tag{23}$$

where $\dot{\varphi}^\ell_{\mathbf{k}} = \partial\varphi_{\mathbf{k}}/\partial k_\ell$. Thus the so-called quantum distance $ds^2 = \sum_{\ell\ell'} g^{\mathbf{k}}_{\ell\ell'} dk_\ell dk_{\ell'} \sim d\theta_{\mathbf{k}} d\theta_{\mathbf{k}} + \sin^2\theta_{\mathbf{k}} d\varphi_{\mathbf{k}} d\varphi_{\mathbf{k}}$ clearly illustrates that $g^{\mathbf{k}}_{\ell\ell'}$ corresponds to nothing but to the natural Fubini-Study metric on the Bloch sphere with radius $r = 1/2$ [20]. When $\theta_{\mathbf{k}} \in \{0, \pi\}$ or $\dot{\varphi}^\ell_{\mathbf{k}} = 0$, Eq. (23) reduces to $\dot{\theta}^\ell_{\mathbf{k}}\dot{\theta}^{\ell'}_{\mathbf{k}}/4$, which gives precisely the interband contribution to the effective-mass tensor in Eq. (22) up to a prefactor, i.e.,

$$\left(M^{-1}_{\text{eff}}\right)_{\ell\ell'} = (M^{-1})_{\ell\ell'} + 2n_0(\mathcal{U} - \mathcal{V})g^{\mathbf{k_c}}_{\ell\ell'}. \tag{24}$$

However, since the former two cases have trivial geometry, here we concentrate only on the latter case (i.e., $\dot{\varphi}^\ell_{\mathbf{k_c}} = 0$) requiring that either (i) $d^x_{\mathbf{k_c}} = 0 = \dot{d}^{x,\ell}_{\mathbf{k_c}}$ or (ii) $d^y_{\mathbf{k_c}} = 0 = \dot{d}^{y,\ell}_{\mathbf{k_c}}$, but not both simultaneously, where $\dot{d}^{i,\ell}_{\mathbf{k_c}} = (\partial d^i_{\mathbf{k}}/\partial k_\ell)_{\mathbf{k}=\mathbf{k_c}}$. Note that Eq. (21) is derived for a uniformly condensed Bose gas; i.e., it assumes $d^z_{\mathbf{k_c}} = 0$ as well. Thus the latter possibility (ii) is in complete agreement with the previous works [11–14]: the quantum-metric tensor appears in the Bogoliubov spectrum of a uniformly condensed Bose gas when all of the $\mathbf{k}$ states for the lowest-lying Bloch band $|-, \mathbf{k}\rangle$ admit real representation in the Brillouin zone, i.e., when $d^y_{\mathbf{k}} = 0$ for every $\mathbf{k}$. When this condition holds, it automatically guarantees that $\dot{d}^{y,\ell}_{\mathbf{k}} = 0$ for every $\mathbf{k}$ as well, leading eventually to $g^{\mathbf{k_c}}_{\ell\ell'} = \dot{d}^{z,\ell}_{\mathbf{k_c}}\dot{d}^{z,\ell'}_{\mathbf{k_c}}/(2d^x_{\mathbf{k_c}})^2$ for case (ii). Similarly we find $g^{\mathbf{k_c}}_{\ell\ell'} = \dot{d}^{z,\ell}_{\mathbf{k_c}}\dot{d}^{z,\ell'}_{\mathbf{k_c}}/(2d^y_{\mathbf{k_c}})^2$ for the former possibility (i) when $d^x_{\mathbf{k}} = 0$ for every $\mathbf{k}$.

In the particular case when the Bloch Hamiltonian exhibits time-reversal symmetry [i.e., when the Hamiltonian matrix given in Eq. (2) satisfies $h_{0\mathbf{k}} = h^*_{0,-\mathbf{k}}$ or simply $d^x_{\mathbf{k}} = d^x_{-\mathbf{k}}$, $d^y_{\mathbf{k}} = -d^y_{-\mathbf{k}}$, and $d^z_{\mathbf{k}} = d^z_{-\mathbf{k}}$], a uniformly condensed Bose gas at the zero-momentum Bloch state (i.e., when $\mathbf{k_c} = \mathbf{0}$ and $\theta_{\mathbf{0}} = \pi/2$) has a trivial geometric contribution to the effective-mass tensor of the superfluid carriers. This is because $\dot{\theta}^\ell_{\mathbf{0}} = 0$ when $d^z_{\mathbf{0}} = 0 = \dot{d}^{z,\ell}_{\mathbf{0}}$. See also Appendix A. Thus, when the time-reversal symmetry manifests, a uniformly condensed Bose gas must occur at a finite momentum Bloch state (i.e., $\mathbf{k_c} \neq \mathbf{0}$ and $\theta_{\mathbf{k_c}} = \pi/2$) in order for a nontrivial geometric contribution to appear. Such a situation can only be realized if there exists a degeneracy in the single-particle ground state, e.g., in the presence of a spin-orbit coupling or in a flat Bloch band. The former possibility has recently been addressed in full detail [14]. However, in the latter possibility [16], since the bare inverse band-mass tensor $(M^{-1})_{\ell\ell'}$ necessarily vanishes in Eq. (24), the low-energy Bogoliubov modes are determined entirely by a particular value of the quantum-metric tensor, i.e., by $g^{\mathbf{k_c}}_{\ell\ell'}$ [11–13]. Example models are given in Appendix B.

D. Comparison with the Fermi superfluids

Typically the building blocks for the many-body problem in Fermi superfluids can be found in the two-body problem, and it turns out the quantum-geometric effects are already apparent in this exactly solvable limit [8,9]. For instance, in the presence of time-reversal symmetry and under the condition of uniform pairing on all sublattices within a unit cell, the inverse effective-mass tensor for the lowest-lying two-body bound-state band has a quantum-geometric contribution that is controlled precisely by the quantum-metric tensor of the underlying Bloch states. The exact relation is in fact a $\mathbf{k}$-space sum over a few terms that can be associated with either the intraband or the interband processes, where the band-resolved quantum-metric tensor appears in the latter with some additional energy factors [8,9]. It turns out the many-body problem is quite similar to the two-body one: the inverse effective-mass tensor of the superfluid carriers (i.e., the Cooper pairs) also has a quantum-geometric contribution originating from the interband processes [3,4,6]. This finding further suggests that all of the superfluid properties that depend on the pair mass must have some quantum-geometric contribution, including but not limited to the superfluid weight/density (i.e., superfluid stiffness) and low-energy collective excitations (i.e., Goldstone modes) [1–7].

IV. CONCLUSION

In summary here we considered a weakly interacting Bose gas that is described by a generic two-band Bose-Hubbard model, and we derived its Bogoliubov spectrum for the low-energy quasiparticle excitations. We showed that the interband processes that are induced by the interactions give rise to a quantum-geometric contribution and dress the effective-mass tensor of the superfluid carriers. When the BEC occurs uniformly within a unit cell (i.e., equal condensate filling on both sublattices), we also related the geometric contribution to the quantum-metric tensor of the Bloch states, which is nothing but the natural Fubini-Study metric on the Bloch sphere. Thus, in the particular case when the bare band-mass tensor vanishes (e.g., in a flat Bloch band), the energetic stability of the Bogoliubov modes, and therefore the BEC itself, is guaranteed by a finite quantum-geometric contribution. This shows that the previous results are immune to the presence of nonlocal interactions [11–13].

Similar to the Bogoliubov spectrum and superfluid weight and density, we expect that all of the superfluid properties that depend on the effective carrier mass to have analogous quantum-geometric contributions. These contributions can be distinguished by their linear dependence on the interactions and are well-worthy of further research and exploration. In particular, since our formalism is based on the Bogoliubov approximation, our analytical expressions are not valid away from the weakly interacting limit. As the interactions become stronger, we expect the intraband contribution coming from the upper Bloch band to affect the effective-mass tensor, especially when the interaction energy becomes comparable to the total bandwidth of the single-particle spectrum. Note that such a contribution plays a negligible role in the weakly interacting limit, thanks to the band gap that is protecting the ground state.

Note added. Recently, Ref. [13] appeared in the preprint server, where the speed of Bogoliubov sound is calculated up to second-order in the interactions for a flat-band BEC.

ACKNOWLEDGMENT

The author acknowledges funding from TÜBİTAK.

APPENDIX A: BENCHMARK WITH THE EXTENDED BOSE-HUBBARD MODEL

Here we specifically consider the extended Bose-Hubbard model with only nearest-neighbor hopping $t > 0$, on-site repulsion U, and nearest-neighbor repulsion V, and we assume time-reversal symmetry. In addition we also suppose that the BEC occurs at the zero-momentum Bloch state, and it is uniform in a unit cell, i.e., $d^z_{\mathbf{k_c}=\mathbf{0}} = 0$. Under these conditions, there is no geometric contribution to the low-energy Bogoliubov modes as discussed in Sec. III C. As an illustration, we calculate the Bogoliubov spectrum given in Eq. (12) and find

$$E_{s\mathbf{q}} = s\sqrt{(\varepsilon_{-,\mathbf{q}} - \varepsilon_{-,\mathbf{0}})(\varepsilon_{-,\mathbf{q}} - \varepsilon_{-,\mathbf{0}} + n_0\mathcal{U} + I_\mathbf{q})}, \quad \text{(A1)}$$

$$I_\mathbf{q} = -2n_0 u_\mathbf{q} v_\mathbf{q} \mathrm{Re}[U_{AB}(\mathbf{q})e^{i\varphi_\mathbf{q}}], \quad \text{(A2)}$$

where we use the chemical potential $\mu = \varepsilon_{-,\mathbf{0}} + n_0(\mathcal{U} + \mathcal{V})/2$ and $\varphi_{-\mathbf{q}} = -\varphi_\mathbf{q}$. Here $\mathcal{U} = U$, $\mathcal{V} = zV$, and z is the coordination number, e.g., $z = \{3, 4, 6\}$ for honeycomb, square, and triangular lattices. Note that the contribution from the nearest-neighbor interactions can also be written as $I_\mathbf{q} = n_0(V/t)[(d^x_\mathbf{q})^2 + (d^y_\mathbf{q})^2]/d_\mathbf{q}$, and Eq. (A1) reproduces the usual result when $V = 0$.

As a nontrivial example, let us consider a square lattice with lattice spacing a and describe it with a unit cell that contains a two-point basis, i.e., treat it like a bipartite checkerboard lattice. Then its single-particle spectrum is characterized by $d^x_\mathbf{k} = -2t[\cos(k_x a) + \cos(k_y a)]$ and $d^y_\mathbf{k} = d^z_\mathbf{k} = d^0_\mathbf{k} = 0$. In this case Eq. (A1) reproduces the known result [17] in the reduced Brillouin zone, i.e., in a square region bounded by $|k_x| + |k_y| = \pi/a$ since the lattice period is doubled in both the x and y directions, in which $\varphi_\mathbf{k} = \pi$ for every $\mathbf{k}$. Note that their condensate filling is defined per lattice site, i.e., $n_0 = 2\nu$. Thus our projected Hamiltonian and its Bogoliubov theory are quantitatively accurate in describing the low-energy physics. Furthermore it can be shown (for sufficiently large V) that Eq. (A1) develops a roton minimum at the edges of the reduced Brillouin zone. Then by setting, e.g., $(\partial^2 E_{-,\mathbf{q}}/\partial q_x^2)_{\mathbf{q}=(0,\pi/a)} \geqslant 0$ at one of the corners, we find that the roton minimum occurs if $n_0(\mathcal{V} - \mathcal{U}) \geqslant 8t$. This condition coincides precisely with the threshold for the dynamical instability which signals the superfluid-to-supersolid phase transition [17].

APPENDIX B: EXAMPLE MODELS WITH NONTRIVIAL GEOMETRY

The Mielke checkerboard model is one of the simplest two-band lattice models that exhibit a flat band in two dimensions. See the Supplemental material of Ref. [21] for a realistic proposal of its implementation using optical-lattice potentials. Within our reciprocal-space convention, the single-particle spectrum in such a lattice is described by $d^0_\mathbf{k} = 2t\cos(k_x a)\cos(k_y a)$, $d^x_\mathbf{k} = 2t\cos(k_x a) + 2t\cos(k_y a)$, and $d^z_\mathbf{k} = 2t\sin(k_x a)\sin(k_y a)$, leading to a flat lower band $\varepsilon_{-,\mathbf{k}} = -2t$ and a dispersive upper band $\varepsilon_{+,\mathbf{k}} = 2t + 4t\cos(k_x a)\cos(k_y a)$. Thus the resultant Bloch bands touch at the four corners of the Brillouin zone, i.e., at $\mathbf{k} = \{(\pm\pi/a, 0), (0, \pm\pi/a)\}$. Setting $d^z_\mathbf{k} = 0$ shows that there exists a continuous subset of flat-band states that favor uniform condensation on sublattices A and B and, hence, is expected to minimize the condensation energy. Since its flat-band states also admit real representation for every $\mathbf{k}$, Eqs. (22) and (24) directly apply to this model. In addition, the model also exhibits time-reversal symmetry, and therefore a finite $\mathbf{k_c}$ guarantees a nontrivial geometric contribution.

Other two-band models that feature a flat band in two dimensions include checkerboard I, II, and III lattices [22]. For instance, the latter model is described by $d^0_\mathbf{k} = 7t/2 + t\cos(k_x a) + 2t\cos(k_y a)$, $d^x_\mathbf{k} = -2t - 2t\cos(k_x a) - t\cos(k_y a) - 2t\cos(k_x a + k_y a)$, $d^y_\mathbf{k} = -2t\sin(k_x a) - t\sin(k_y a) - 2t\sin(k_x a + k_y a)$, and $d^z_\mathbf{k} = 3t/2 - t\cos(k_x a) + 2t\cos(k_y a)$, and it leads to a flat lower band $\varepsilon_{-,\mathbf{k}} = 0$ that is gapped from the dispersive upper band. The minimum band gap occurs at the four corners of the Brillouin zone, i.e., at $\mathbf{k} = \{(\pm\pi/a, 0), (0, \pm\pi/a)\}$. Setting again $d^z_{\mathbf{k}} = 0$ shows that there exists a continuous subset of flat-band states that favor uniform condensation on sublattices A and B and, hence, minimize the condensation energy [12]. While this model also exhibits time-reversal symmetry, its flat-band states do not admit real representation for every $\mathbf{k}$. Thus only Eq. (22) applies to this model. See Ref. [12] for a detailed analysis of this particular model and its numerical illustration.

[1] S. Peotta and P. Törmä, Superfluidity in topologically nontrivial flat bands, Nat. Commun. **6**, 8944 (2015).

[2] L. Liang, T. I. Vanhala, S. Peotta, T. Siro, A. Harju, and P. Törmä, Band geometry, Berry curvature, and superfluid weight, Phys. Rev. B **95**, 024515 (2017).

[3] K.-E. Huhtinen, J. Herzog-Arbeitman, A. Chew, B. A. Bernevig, and P. Törmä, Revisiting flat band superconductivity: Dependence on minimal quantum metric and band touchings, Phys. Rev. B **106**, 014518 (2022).

[4] J. Herzog-Arbeitman, A. Chew, K.-E. Huhtinen, P. Törmä, and B. A. Bernevig, Many-body superconductivity in topological flat bands, arXiv:2209.00007.

[5] S. M. Chan, B. Grémaud, and G. G. Batrouni, Designer flat bands: Topology and enhancement of superconductivity, Phys. Rev. B **106**, 104514 (2022).

[6] M. Iskin, Quantum-metric contribution to the pair mass in spin-orbit-coupled Fermi superfluids, Phys. Rev. A **97**, 033625 (2018).

[7] M. Iskin, Collective excitations of a BCS superfluid in the presence of two sublattices, Phys. Rev. A **101**, 053631 (2020).

[8] M. Iskin, Two-body problem in a multiband lattice and the role of quantum geometry, Phys. Rev. A **103**, 053311 (2021).

[9] M. Iskin, Effective-mass tensor of the two-body bound states and the quantum-metric tensor of the underlying Bloch states in multiband lattices, Phys. Rev. A **105**, 023312 (2022).

[10] P. Törmä, L. Liang, and S. Peotta, Quantum metric and effective mass of a two-body bound state in a flat band, Phys. Rev. B **98**, 220511(R) (2018).

[11] A. Julku, G. M. Bruun, and P. Törmä, Quantum Geometry and Flat Band Bose-Einstein Condensation, Phys. Rev. Lett. **127**, 170404 (2021).

[12] A. Julku, G. M. Bruun, and P. Törmä, Excitations of a Bose-Einstein condensate and the quantum geometry of a flat band, Phys. Rev. B **104**, 144507 (2021).

[13] A. Julku, G. Salerno, and P. Törmä, Supefluidity of flat band Bose-Einstein condensates revisited, arXiv:2210.11906.

[14] A. L. Subaşı and M. Iskin, Quantum-geometric perspective on spin-orbit-coupled Bose superfluids, Phys. Rev. A **105**, 023301 (2022).

[15] M. Iskin, Exposing the quantum geometry of spin-orbit-coupled Fermi superfluids, Phys. Rev. A **97**, 063625 (2018).

[16] Y.-Z. You, Z. Chen, X.-Q. Sun, and H. Zhai, Superfluidity of Bosons in Kagome Lattices with Frustration, Phys. Rev. Lett. **109**, 265302 (2012).

[17] I. Danshita and C. A. R. Sá de Melo, Stability of Superfluid and Supersolid Phases of Dipolar Bosons in Optical Lattices, Phys. Rev. Lett. **103**, 225301 (2009).

[18] M. Iskin, Route to supersolidity for the extended Bose-Hubbard model, Phys. Rev. A **83**, 051606(R) (2011).

[19] R. Resta, The insulating state of matter: a geometrical theory, Eur. Phys. J. B **79**, 121 (2011).

[20] Note that the imaginary part of the quantum-geometric tensor $\Omega^{\pm,\mathbf{k}}_{\ell\ell'} = \pm\frac{\sin\theta_{\mathbf{k}}}{4}(\dot{\theta}^{\ell}_{\mathbf{k}}\dot{\varphi}^{\ell'}_{\mathbf{k}} + \dot{\theta}^{\ell'}_{\mathbf{k}}\dot{\varphi}^{\ell}_{\mathbf{k}})$ is directly related to the Berry curvature $F^{\pm}_{\theta_{\mathbf{k}}\varphi_{\mathbf{k}}} = \mp\frac{1}{2}\sin\theta_{\mathbf{k}}$.

[21] G. Montambaux, L.-K. Lim, J.-N. Fuchs, and F. Piéchon, Winding Vector: How to Annihilate Two Dirac Points with the Same Charge, Phys. Rev. Lett. **121**, 256402 (2018).

[22] J.-W. Rhim and B.-J. Yang, Classification of flat bands according to the band-crossing singularity of Bloch wave functions, Phys. Rev. B **99**, 045107 (2019).