Bardeen-Cooper-Schrieffer–type pairing in a spin-$\frac{1}{2}$ Bose gas with spin-orbit coupling

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We apply the functional path-integral approach to analyze how the presence of a spin-orbit coupling (SOC) affects the basic properties of a BCS-type paired state in a two-component Bose gas. In addition to a mean-field theory that is based on the saddle-point approximation for the intercomponent pairing, we derive a Ginzburg-Landau theory by including the Gaussian fluctuations on top, and use them to reveal the crucial roles played by the momentum-space structure of an arbitrary SOC field in the stability of the paired state at finite temperatures. For this purpose, we calculate the critical transition temperature for the formation of paired bosons, and that of the gapless quasiparticle excitations for a broad range of interaction and SOC strengths. In support of our results for the many-body problem, we also benchmark our numerical calculations against the analytically tractable limits, and provide a full account of the two-body limit including its nonvanishing binding energy for arbitrarily weak interactions and the anisotropic effective mass tensor.

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I. INTRODUCTION

When atomic fermions transform into molecular bosons by way of many-body pairing, the mechanical stability of the paired state is inherently enforced by the Pauli exclusion principle, i.e., through a Hartree shift of the chemical potential by inducing a pairwise interaction that is effectively weak and repulsive. This intrinsic stability is what lies behind the long-sought realization of the so-called BCS-BEC crossover, when a two-component Fermi gas is magnetically swept across a Feshbach resonance [1]. Having witnessed more than a decade of tremendous successes since their creation, the ultracold Fermi gases has become a thriving field in modern quantum physics as it keeps enriching its toolbox with a wide range of applications for the strongly correlated phenomena in a much broader context [2,3].

In comparison, the analogous evolution from the BCS-type many-body paired state of atomic bosons to the BEC of molecular bosons is to a great extent an uncharted territory in a Bose gas. Despite many theoretical attempts dating back more than half a century [4–11], the crossover studies have been hindered by the natural tendency of paired state to a mechanical collapse in the lack of a bosonic counterpart for the exclusion principle. When a spinless Bose gas of atoms is magnetically swept across a Feshbach resonance, the lifetime of the resultant molecules turned out to be too short for reaching an equilibrium state with a molecular BEC [12–15]. To overcome this difficulty, it is quite clear that one needs to search for exotic Bose systems that may exhibit enhanced stability for the many-body pairing. For instance, bosonic particles with an internal spin structure was introduced by Noziéres and Saint James as an alternate [7], and thoroughly analyzed for the spin-1 case.

Motivated by the recent creations of two-component quantum gases with two-dimensional SOCs [16–19], here we revisit this old-standing problem in a so-called spin-$\frac{1}{2}$ Bose gas. Having a dilute Bose gas with short-ranged density-density interactions in mind, we consider an intercomponent attraction $U_{\uparrow\downarrow} = U_{\downarrow\uparrow} = -g < 0$, and analyze how the presence of a SOC affects the resultant pairing correlations [20,21]. The mechanical collapse is counteracted by the Hartree terms arising from the intracomponent repulsions [7,21–23]. Then, assuming that the instability towards a BCS-like paired state is favored against the competing states, e.g., collapse, fragmentation, phase separation, etc. [4–11,20,21], one may treat the intercomponent attraction through a close analogy with the theory of paired fermions [2,3]. For this purpose, we apply the functional path-integral approach, and derive a mean-field theory that is based on the saddle-point approximation for pairing, and then a Ginzburg-Landau theory by including the Gaussian fluctuations on top. Our analysis suggests that, while the SOC has a minor role in the strong-interaction limit where the ground state at zero temperature is a BEC of paired bosons, increasing its strength in the weak-interaction limit may allow for the creation of a paired state at much lower temperatures. We also provide a full account of the two-body problem including its nonvanishing binding energy for arbitrarily weak interactions and the anisotropic effective mass tensor.

II. THREE-DIMENSIONAL SPIN-$\frac{1}{2}$ BOSE GAS WITH AN ARBITRARY SOC

We are interested in a two-component Bose gas that is described by the many-body Hamiltonian

$$H = \sum_{ab\mathbf{k}} c^\dagger_{a\mathbf{k}}(\xi_\mathbf{k}\sigma_0 + \mathbf{S}_\mathbf{k}\cdot\boldsymbol{\sigma})_{ab}c_{b\mathbf{k}} + \frac{1}{2}\sum_{ab\mathbf{k}\mathbf{k}'\mathbf{q}} U_{ab}c^\dagger_{a,\mathbf{k}+\mathbf{q}/2}c^\dagger_{b,-\mathbf{k}+\mathbf{q}/2}c_{b,-\mathbf{k}'+\mathbf{q}/2}c_{a,\mathbf{k}'+\mathbf{q}/2}, \quad (1)$$

where the wave vector $\mathbf{k} = (k_x, k_y, k_z)$ labels the momentum eigenstates, and the spin $a \in \{\uparrow, \downarrow\}$ labels the atomic components in such a way that $c^\dagger_{a\mathbf{k}}$ creates a spin-a boson with momentum $\mathbf{k}$ (in units of $\hbar \to 1$). Assuming the components are population balanced, $\xi_\mathbf{k} = \epsilon_\mathbf{k} - \mu$ includes the parabolic dispersion $\epsilon_\mathbf{k} = k^2/(2m)$ of the particles in free space and their chemical potential $\mu < 0$ [22,23]. In addition, σ_0 is a 2×2 unit matrix, $\mathbf{S}_\mathbf{k} = (S^x_\mathbf{k}, S^y_\mathbf{k}, S^z_\mathbf{k})$ is a SOC field whose components $S^i_\mathbf{k} = \alpha_i k_i$ are controlled independently by the strengths $\alpha_i \geqslant 0$, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli spin matrices. Below it is called an XYZ or a Weyl SOC if $\alpha_x = \alpha_y = \alpha_z = \alpha$ is isotropic in the entire $\mathbf{k}$ space, an XY or a Rashba SOC if $\alpha_x = \alpha_y = \alpha$ is isotropic in $k_x k_y$ plane with $\alpha_z = 0$, and a YZ SOC if $\alpha_y = \alpha_z = \alpha$ is isotropic in $k_y k_z$ plane with $\alpha_x = 0$.

III. MEAN-FIELD THEORY FOR THE INTERCOMPONENT PAIRING

After a straightforward algebra, the saddle-point contribution Ω_0 to the thermodynamic potential can be written as $\Omega_0 = A_0 + (T/2)\mathrm{Tr} \sum_k \ln[G_0^{-1}(k)/T]$. Here, $A_0 = |\Delta_0|^2/g - \sum_\mathbf{k} \xi_\mathbf{k}$ with the complex number Δ_0 corresponding to the mean-field order parameter for the stationary pairs, and determined by the thermal average $\Delta_\mathbf{q} = -g\sum_\mathbf{k} \langle c_{\downarrow,-\mathbf{k}+\mathbf{q}/2} c_{\uparrow,\mathbf{k}+\mathbf{q}/2}\rangle$ in the $\mathbf{q} \to \mathbf{0}$ limit. In addition, T is the temperature with $k_B \to 1$ the Boltzmann constant, Tr is the trace, and k denotes a combined summation index for $(\mathbf{k}, i\omega_\ell)$ where $\omega_\ell = 2\pi T \ell$ is the bosonic Matsubara frequency for the particles with ℓ an integer. Furthermore,

$$G_0^{-1} = \begin{bmatrix} (i\omega_\ell + \xi_\mathbf{k})\sigma_0 + \mathbf{S}_\mathbf{k}\cdot\boldsymbol{\sigma} & \Delta_0\sigma_x \\ \Delta_0^*\sigma_x & (-i\omega_\ell + \xi_\mathbf{k})\sigma_0 - \mathbf{S}_\mathbf{k}\cdot\boldsymbol{\sigma}^* \end{bmatrix}$$

corresponds to the inverse Green's function associated with the Nambu spinor $\psi_k^\dagger = (c^\dagger_{\uparrow k}, c^\dagger_{\downarrow k}, c_{\uparrow,-k}, c_{\downarrow,-k})$. A compact way to express $\Omega_0 = A_0 + (T/2)\sum_{ss'k} \ln[(i\omega_\ell + s' E_{s\mathbf{k}})/T]$ is through the quasiparticle energies $E_{s\mathbf{k}} = \sqrt{\xi_\mathbf{k}^2 - |\Delta_0|^2 + S_\mathbf{k}^2 + 2sB_\mathbf{k}}$, where $s \in \{+,-\}$, $S_\mathbf{k} = [(S^\perp_\mathbf{k})^2 + (S^z_\mathbf{k})^2]^{1/2}$ is the strength of the SOC field with $S^\perp_\mathbf{k} = [(S^x_\mathbf{k})^2 + (S^y_\mathbf{k})^2]^{1/2}$, and $B_\mathbf{k} = [\xi_\mathbf{k}^2 S_\mathbf{k}^2 - |\Delta_0|^2 (S^\perp_\mathbf{k})^2]^{1/2}$. We note that the quasiparticle energies reduce to $E_{s\mathbf{k}} = \sqrt{\xi_\mathbf{k}^2 - |\Delta_0|^2} + sS^\perp_\mathbf{k}$ when $\alpha_z = 0$, $E_{s\mathbf{k}} = \sqrt{(\xi_\mathbf{k} + sS^z_\mathbf{k})^2 - |\Delta_0|^2}$ when $\alpha_x = \alpha_y = 0$, and $E_{s\mathbf{k}} = \xi_{s\mathbf{k}} = \xi_\mathbf{k} + sS_\mathbf{k}$ when $\Delta_0 \to 0$.

The mean-field self-consistency equations for Δ_0 and μ are determined, respectively, by setting $\partial\Omega_0/\partial|\Delta_0| = 0$ and $N_0 = -\partial\Omega_0/\partial\mu$, leading to either $\Delta_0 = 0$ or

$$\frac{1}{g} = -\frac{1}{2}\sum_{s\mathbf{k}} \frac{\partial E_{s\mathbf{k}}}{\partial|\Delta_0|^2} \mathcal{X}_{s\mathbf{k}}, \tag{2}$$

$$N_0 = -\frac{1}{2}\sum_{s\mathbf{k}} \left(1 + \frac{\partial E_{s\mathbf{k}}}{\partial\mu}\mathcal{X}_{s\mathbf{k}}\right). \tag{3}$$

Here, N_0 is the thermal average number of particles at the mean-field level, $\partial E_{s\mathbf{k}}/\partial|\Delta_0| = -|\Delta_0|[1 + s(S^\perp_\mathbf{k})^2/B_\mathbf{k}]/E_{s\mathbf{k}}$, $\partial E_{s\mathbf{k}}/\partial\mu = -\xi_\mathbf{k}(1 + sS_\mathbf{k}^2/B_\mathbf{k})/E_{s\mathbf{k}}$, and $\mathcal{X}_{s\mathbf{k}} = \coth[E_{s\mathbf{k}}/(2T)]$ is a thermal factor. Equations (2) and (3) follow from a Matsubara summation of the form $T\sum_\ell 1/(i\omega_\ell - x) = -n_B(x)$, where $n_B(x) = 1/(e^{x/T} - 1)$ is the Bose-Einstein distribution with $n_B(x) + n_B(-x) = -1$ and $\coth[x/(2T)] = 1 + 2n_B(x)$. It can be readily verified that all of these expressions recover the known counterparts in the absence of a SOC when $S_\mathbf{k} \to 0$ [8,10,11]. In addition, since a unidirectional SOC field in $\mathbf{k}$ space (e.g., $S_\mathbf{k} = |S^i_\mathbf{k}|$ for any $i \in \{x, y, z\}$) can be trivially gauged or integrated away from the self-consistency equations, it is identical to the $S_\mathbf{k} \to 0$ case up to an energy offset in μ.

Following the standard prescription for the BCS-BEC crossover problem [3], we substitute the bare interaction strength g between the $\uparrow$ and $\downarrow$ bosons with the associated s-wave scattering length a_s in vacuum through the relation $1/g = -mV/(4\pi a_s) + \sum_\mathbf{k} 1/(2\epsilon_\mathbf{k})$, where V is the volume. In addition, we define a length scale k_0 through an analogy with the number equation $N_0 = k_0^3 V/(3\pi^2)$ of a free Fermi gas at $T = 0$, along with the corresponding energy scale $\epsilon_0 = k_0^2/(2m)$. Then, we solve Eqs. (2) and (3) for the saddle-point parameters $|\Delta_0|/\epsilon_0$ and μ/ϵ_0, and analyze their stability as functions of $1/(k_0 a_s)$, T/ϵ_0, and $m\alpha/k_0$. The resultant phase diagrams are presented in Figs. 1 and 2, and they are constructed as follows.

A. Critical pairing transition temperature

Recalling that the thermodynamic stability of the paired state that is described by this mean-field theory requires a nonzero order parameter, we introduce an upper bound on T that is based on the critical pairing transition temperature T_p, below which $\Delta_0 > 0$. Thus, by setting $\Delta_0 \to 0^+$ in Eqs. (2) and (3), we find $1/g = \sum_{s\mathbf{k}} \mathcal{X}^p_{s\mathbf{k}}[1 - s(S^z_\mathbf{k})^2/(\xi_{s\mathbf{k}} S_\mathbf{k})]/(4\xi_\mathbf{k})$ for the T_p equation, and $N_0 = \sum_{s\mathbf{k}} n_B^p(\xi_{s\mathbf{k}})$ for the number equation, where $\mathcal{X}^p_{s\mathbf{k}} = \coth[\xi_{s\mathbf{k}}/(2T_p)]$. In addition, by requiring $\xi_{s\mathbf{k}} > 0$ in the entire $\mathbf{k}$ space for the normal state, we find that $|\mu| > m\alpha_m^2/2$ with $\alpha_m = \max\{\alpha_x, \alpha_y, \alpha_x\}$ corresponds to a lower bound on the stability of the mean-field T_p equation. $\Delta_0 = 0$ below this bound, opening a window for the BEC of unpaired bosons.

In Fig. 1, our numerical calculations show that T_p/ϵ_0 saturates to 0.5 in the absence of a SOC in the weak-interaction limit when $g \to 0^+$ or $1/(k_0 a_s) \to -\infty$. This is in perfect agreement with our analytical derivations. In addition, the mere effect of having a finite Weyl or YZ SOC on T_p/ϵ_0 is seen to be the lowering of the saturation value in the strong-interaction limit when $1/(k_0 a_s) \to +\infty$. It is pleasing to see that this α-dependent effect gradually fades away with increasing $1/(k_0 a_s)$, and T_p/ϵ_0 eventually conforms to a SOC-independent growth in the opposite $1/(k_0 a_s) \to +\infty$ limit. In sharp contrast, Fig. 1 shows that having a finite Rashba SOC has a rather dramatic effect on T_p/ϵ_0, and that the self-consistency equations do not yield a convergent solution once $1/(k_0 a_s)$ is below an α-dependent value. Thus the pairing transition is discontinuous, and the precise location of the jumps in T_g/ϵ_0 are determined by $|\mu| \to m\alpha^2/2$, signaling the violation of the $\xi_{s\mathbf{k}} > 0$ condition for the stability of the normal state.

B. Critical gapless transition temperature

In addition, for the dynamical stability of the fully paired state, it is necessary to restrict the self-consistent solutions to the parameter regime where the quasiparticle energies are real and positive in the entire $\mathbf{k}$ space. Note that $E_{s\mathbf{k}} \leqslant 0$ may indicate a competition between a paired state and an unpaired one [10,11]. Thus, by imposing the condition $E_{+,\mathbf{k}}E_{-,\mathbf{k}} = 0$, we

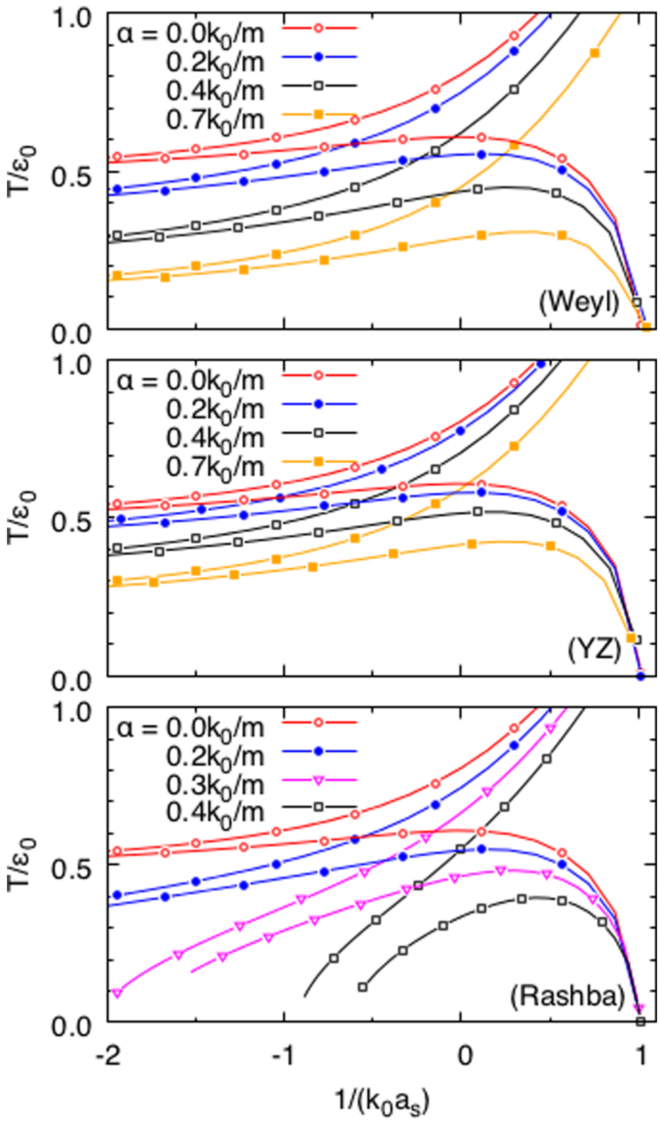

FIG. 1. Finite temperature phase diagrams are constructed for the Weyl (upper panel), YZ (middle), and Rashba (lower) SOC fields. In each of these panels, we choose a set of SOC strengths α, and vary the scattering length a_s. For a given α, the BCS-type paired state is bounded by an upper and a lower curve, corresponding, respectively, to the critical transition temperature for the formation of pairs (T_p), and that of the gapless quasiparticle excitations (T_g).

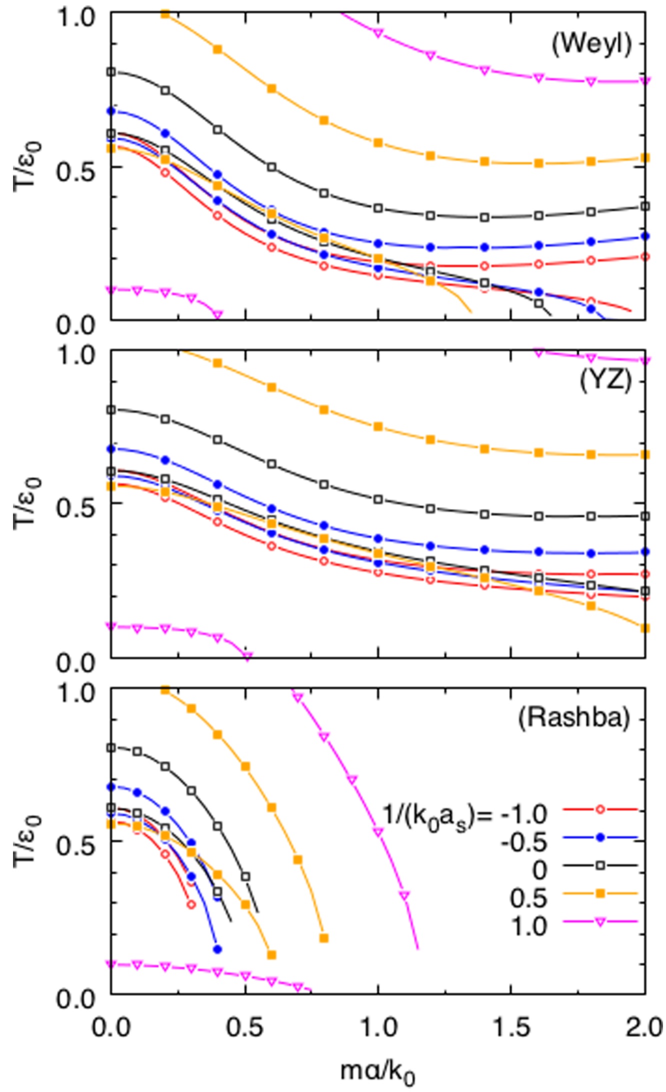

FIG. 2. Finite temperature phase diagrams are constructed for the Weyl (upper panel), YZ (middle), and Rashba (lower) SOC fields. In each of these panels, we choose a set of scattering lengths a_s and vary the SOC strength α. For a given a_s, the BCS-type paired state is bounded by an upper and a lower curve, corresponding, respectively, to the critical transition temperature for the formation of pairs (T_p) and that of the gapless quasiparticle excitations (T_g).

find that the excitations are gapless in those **k**-space regions satisfying $(\xi_{\mathbf{k}} + |\Delta_0| + S^z_{\mathbf{k}})(\xi_{\mathbf{k}} - |\Delta_0| - S^z_{\mathbf{k}}) = (S^\perp_{\mathbf{k}})^2$. Then, by analyzing the gradient of $|\mu|$ in **k** space, we conclude that the gapless region is bounded by a minimum value of $|\mu|$ determined by $|\mu| = m\alpha_z^2/2 + |\Delta_0|$ when $|\Delta_0| > m|\alpha_m^2 - \alpha_z^2|$, and by $|\mu| = m\alpha_m^2/2 + |\Delta_0|^2/(2m|\alpha_m^2 - \alpha_z^2|)$ when $|\Delta_0| \leqslant m|\alpha_m^2 - \alpha_z^2|$.

In order to identify this instability [10,11], here we introduce a lower bound on T that is based on the critical gapless transition temperature T_g, above which $E_{s\mathbf{k}} > 0$. For instance, noting that the gapless transition condition reduces to $\mu = -|\Delta_0|$ in the absence of a SOC, one can analytically determine the precise location of $T_g \to 0^+$. We find $|\mu| = \pi^2/(16ma_s^2)$ from the order parameter equation, and $|\mu| = 2^{1/3}\epsilon_0$ from the number equation, leading to a critical point $1/(k_0a_s) = 2^{5/3}/\pi \approx 1.0106$ that is in perfect agreement with our numerical calculations. Independent of the **k**-space structure, Fig. 1 shows that the precise location of the critical point is quite immune to the presence of a weak SOC, and that the ground ($T = 0$) state is stable for $1/(k_0a_s) > 1.0106$ only. However, Fig. 2 shows that stronger SOCs eventually increase the stability of the ground state towards the lower $1/(k_0a_s)$ regions as well. Going away from the critical point, one expects to find $T_g \to T_p$ from below as $\Delta_0 \to 0^+$ in the $1/(k_0a_s) \to -\infty$ limit, and this turns out to be the case for a Weyl or a YZ SOC. In sharp contrast, Figs. 1 and 2 again show peculiar jumps for the Rashba SOC, suggesting a discontinuous transition. We emphasize that these jumps are not numerical artifacts, and their origin can be traced back to the observation that self-consistency equations do not yield a convergent solution with $0 < |\Delta_0| < m\alpha^2$ and, therefore, with $m\alpha^2/2 < |\mu| < m\alpha^2$.

So far the mean-field analysis reveals that the **k**-space structure of a SOC plays a crucial role in the boosted stability of the paired state at low T, but what makes a Rashba SOC distinct from a Weyl or a YZ SOC is yet to be uncovered. To address this question, next we analyze the Gaussian fluctuations of the order parameter around its mean field, and gain more physical insight into the pairing problem.

IV. GAUSSIAN FLUCTUATIONS NEAR T_p

Going beyond the saddle-point approximation for pairing in the $\Delta_0 \to 0$ limit [10,24], the Gaussian-fluctuation contribution Ω_G to the thermodynamic potential can be written as $\Omega_G = \sum_q |\Lambda_q|^2/g - (T/4)\mathrm{Tr}\sum_{kq} G_0(k)\Sigma(q)G_0(k+q)\Sigma(-q)$. Here, $q = (\mathbf{q}, i\nu_\ell)$ is a combined summation index with $\nu_\ell = 2\pi T\ell$ the bosonic Matsubara frequency for the pairs, Λ_q is the spatial and temporal fluctuations of the order parameter around its saddle-point value, and $\Sigma(q) = [\begin{smallmatrix} 0 & \Lambda_q\sigma_x \\ \Lambda^*_{-q}\sigma_x & 0 \end{smallmatrix}]$. A compact way to express $\Omega_G = \sum_q \mathcal{L}_q^{-1}|\Lambda_q|^2$ is through

$$\mathcal{L}_q^{-1} = \frac{1}{g} - \frac{1}{8}\sum_{ss'\mathbf{k}} \frac{\mathcal{X}_{s,\mathbf{k}+\mathbf{q}/2} + \mathcal{X}_{s',-\mathbf{k}+\mathbf{q}/2}}{\xi_{s,\mathbf{k}+\mathbf{q}/2} + \xi_{s',-\mathbf{k}+\mathbf{q}/2} + i\nu_\ell} C_{\mathbf{kq}}^{ss'}, \quad (4)$$

corresponding to the inverse of the fluctuation propagator associated with the pairs, where $\mathcal{X}_{s\mathbf{k}} = \coth[\xi_{s\mathbf{k}}/(2T)]$ is a thermal factor, and $C_{\mathbf{kq}}^{ss'} = 1 + ss'(\mathbf{S}_{\mathbf{k}+\mathbf{q}/2}\cdot\mathbf{S}_{-\mathbf{k}+\mathbf{q}/2} - 2S^z_{\mathbf{k}+\mathbf{q}/2}S^z_{-\mathbf{k}+\mathbf{q}/2})/(S_{\mathbf{k}+\mathbf{q}/2}S_{-\mathbf{k}+\mathbf{q}/2})$.

By expanding the inverse propagator up to second order in the momentum and first order in the frequency of the pairs, we find $\mathcal{L}_q^{-1} = a(T) + \frac{1}{2}\sum_{ij} c_{ij}q_iq_j - d\omega + \cdots$. This is our Ginzburg-Landau theory in disguise [24], whose zeroth-order term $a(T) = \mathcal{L}_0^{-1}$ determines the transition temperature, the second-order kinetic coefficient $c_{ij} = \lim_{q\to(\mathbf{0},0)} \partial^2\mathcal{L}_q^{-1}/(\partial q_i\partial q_j)$ is related to the effective mass tensor $\mathbf{m_p}$ of the pairs, and the first-order coefficient $d\omega \overset{\omega\to 0}{=} \mathcal{L}_0^{-1} - \lim_{q\to(\mathbf{0},-\omega+i0^+)}\mathcal{L}_q^{-1}$ characterizes the dynamical stability or lifetime of the pairs. Note that the zeroth-order term can also be written as $a(T) = \lim_{\Delta_0\to 0}\partial\Omega_0/\partial|\Delta_0|^2$, showing that the Thouless condition $a(T_p) = 0$ reproduces the equation for T_p. Similarly, going beyond the Gaussian fluctuations, the zeroth-order coefficient of the fourth-order fluctuations in $\Lambda(q)$ can be approximated as $b = \lim_{\Delta_0\to 0}\partial^2\Omega_0/\partial(|\Delta_0|^2)^2$. This higher-order coefficient controls the interaction strength $g_p = b/d^2$ between pairs, where $b = \lim_{\Delta_0\to 0}\sum_{s\mathbf{k}}\{[\partial^2 E_{s\mathbf{k}}/\partial(|\Delta_0|^2)^2]\mathcal{X}_{s\mathbf{k}}/2 - (\partial E_{s\mathbf{k}}/\partial|\Delta_0|^2)^2\mathcal{Y}_{s\mathbf{k}}/(4T)\}$ with $\mathcal{Y}_{s\mathbf{k}} = \mathrm{csch}^2[\xi_{s\mathbf{k}}/(2T)]$ an additional thermal factor.

The presence of thermal factors makes the coefficients of the Ginzburg-Landau theory rather cumbersome for the many-body problem, and this holds true even in the absence of a SOC. However, it is possible to circumvent around this complication in the two-body limit and make further analytical progress. For instance, for the two-body binding problem in vacuum where $\mu < 0$ with $|\mu| = \epsilon_b/2 \gg T_p \to 0$, by setting the thermal factors to unity (i.e., $\mathcal{X}_{s\mathbf{k}} \to 1$ for every $s\mathbf{k}$ assuming $\xi_{s\mathbf{k}} > 0$) and introducing $\epsilon_b = i\nu_\ell - 2\mu$ as the binding energy at $T = 0$, we find $\mathcal{L}_{\mathrm{tb}}^{-1} = 1/g - (1/4)\sum_{ss'\mathbf{k}} C_{\mathbf{kq}}^{ss'}/(\epsilon_{s,\mathbf{k}+\mathbf{q}/2} + \epsilon_{s',-\mathbf{k}+\mathbf{q}/2} + \epsilon_b)$. This is the inverse propagator for the two-body bound states, and it can be used to extract both the binding energy ϵ_b and the effective mass tensor $\mathbf{m_p}$ of the pairs as follows.

A. Binding energy of the two-body bound state

By applying the Thouless condition for the two-body problem $a_{\mathrm{tb}}(0) = 0$, we find

$$\frac{1}{g} = \sum_{\mathbf{k}} \frac{(2\epsilon_\mathbf{k} + \epsilon_b)^2 - 4(S_\mathbf{k}^\perp)^2}{(2\epsilon_\mathbf{k} + \epsilon_b)\left[(2\epsilon_\mathbf{k} + \epsilon_b)^2 - 4S_\mathbf{k}^2\right]}, \quad (5)$$

which is analytically tractable in various limits. For instance, when $S_\mathbf{k}^\perp = 0$, Eq. (5) suggests that a bound state with $\epsilon_b = 1/(ma_s^2)$ exists only for $a_s > 0$. This result is not surprising given that a unidirectional SOC field in **k** space (e.g., the remaining SOC component $S_\mathbf{k}^z$) can simply be gauged away even from our many-body mean field. More intriguingly, for a Rashba SOC, Eq. (5) again suggests that a bound state with $\epsilon_b = 1/(ma_s^2)$ exists only for $a_s > 0$, and that the presence of a Rashba SOC does not have any effect on the binding energy. This null result is quite surprising given its nontrivial fermionic counterpart, where a bound state with $\epsilon_b \neq 0$ is known to exist for any a_s no matter how weak g is as long as $g \neq 0$ [25]. However, we note that the intracomponent pairing of bosons with Rashba SOC is similar to the fermion problem [20].

On the contrary, for a Weyl SOC, Eq. (5) suggests that a bound state with $\epsilon_b \neq 0$ exists for any a_s, according to $3/(m\alpha a_s) = 2\sqrt{\epsilon_b/(m\alpha^2)} + \sqrt{\epsilon_b/(m\alpha^2) - 1} - 1/\sqrt{\epsilon_b/(m\alpha^2) - 1}$. This leads to $\epsilon_b = m\alpha^2[1 + (m\alpha a_s/3)^2]$ in the weak-binding limit when $\epsilon_b \to m\alpha^2$ or $1/(m\alpha a_s) \to -\infty$, $\epsilon_b = 2m\alpha^2/\sqrt{3} = 1.1547m\alpha^2$ at unitarity when $|a_s| \to \infty$, and $\epsilon_b = m\alpha^2 + 1/(ma_s^2)$ in the strong-binding limit when $\epsilon_b \gg m\alpha^2$ or $1/(m\alpha a_s) \to +\infty$. Thus our formalism recovers the exact solution of the two-body problem known for a Weyl SOC [21]. Similarly, for a YZ SOC, Eq. (5) again suggests that a bound state with $\epsilon_b \neq 0$ exists for any a_s, according to $2/(m\alpha a_s) = 2\sqrt{\epsilon_b/(m\alpha^2)} - \mathrm{arcsinh}[1/\sqrt{\epsilon_b/(m\alpha^2) - 1}]$. This leads to $\epsilon_b = m\alpha^2[1 + 4e^{4/(m\alpha a_s) - 4}]$ in the $1/(m\alpha a_s) \to -\infty$ limit, $\epsilon_b = 1.06640m\alpha^2$ at unitarity, and $\epsilon_b = m\alpha^2 + 1/(ma_s^2)$ in the $1/(m\alpha a_s) \to +\infty$ limit.

Given these analytical results for the two-body problem, we conclude that it is the coupling between $S_\mathbf{k}^z$ and the other components ($S_\mathbf{k}^x$ and/or $S_\mathbf{k}^y$) that gives rise to a two-body bound state with $\epsilon_b \neq 0$ for any $a_s < 0$ as long as $g \neq 0$. This conclusion clearly sheds some light on the boosted stability of the many-body problem in general, and particularly on Figs. 1 and 2.

B. Effective mass of the two-body bound state

It turns out that the elements $(\mathbf{m_p^{-1}})^{ij} = c_{ij}/d$ of the inverse-effective-mass tensor are also analytically tractable for the two-body problem. For instance, in the weak-binding limit when $1/(m\alpha a_s) \to -\infty$, we note that $S_\mathbf{k}^z \neq 0$ for the existence of a two-body bound state to begin with, and find $c_{ij} \to \sum_{s\mathbf{k}}[\partial^2\xi_{s\mathbf{k}}/(\partial k_i \partial k_j)](S^z_{\mathbf{k}})^2/(16\xi^2_{s\mathbf{k}}S^2_{\mathbf{k}})$ and $d \to \sum_{s\mathbf{k}}(S^z_{\mathbf{k}})^2/(8\xi^2_{s\mathbf{k}}S^2_{\mathbf{k}})$. For a Weyl SOC, setting $\epsilon_b \to m\alpha^2$ in the $1/(m\alpha a_s) \to -\infty$ limit, we find $d = mV\sqrt{m}|\mu|/[12\pi(2|\mu| - m\alpha^2)^{3/2}]$, leading to a diagonal mass tensor with anisotropic elements $m_p^{xx} = m_p^{yy} = 10m$ and $m_p^{zz} = 10m/3$. This result is in sharp contrast with its fermionic counterpart, where $m_p^{xx} = m_p^{yy} = m_p^{zz} = 6m$ is known to be isotropic in the entire space [25]. Similarly, for a YZ SOC, setting again $\epsilon_b \to m\alpha^2$ in the $1/(m\alpha a_s) \to -\infty$ limit, we find $d = mV\sqrt{2m|\mu|}/[16\pi(2|\mu| - m\alpha^2)]$, leading again to a diagonal mass tensor with anisotropic elements $m_p^{xx} = 2m$, $m_p^{yy} = 8m$, and $m_p^{zz} = 8m/3$. This result is again in sharp contrast with its fermionic counterpart, where $m_p^{xx} = 2m$ but $m_p^{yy} = m_p^{zz} = 4m$ is known to be isotropic in yz plane [25]. We note that a Rashba SOC gives rise to an anisotropic mass tensor for the pairs, when the pairing is due to the intracomponent attraction [20].

As a result of this two-body analysis, we conclude that the coupling between $S^z_{\mathbf{k}}$ and the other components ($S^x_{\mathbf{k}}$ and/or $S^y_{\mathbf{k}}$) gives rise to an anisotropic $\mathbf{m_p}$ in general for the many-body bound states [26]. One can show that the SOC-induced anisotropy gradually disappears with increasing ϵ_b, in such a way that $m_p^{xx} = m_p^{yy} = m_p^{zz} = 2m$ is eventually isotropic in space in the strong-binding limit when $1/(m\alpha a_s) \to +\infty$. Note that the effective mass tensor of pairs plays a direct role in their finite T phase diagrams. For instance, the critical BEC temperature T_c of noninteracting pairs is determined by their number equation $N_p = \sum_{\mathbf{k}} n_B^c(\epsilon_{p\mathbf{k}})$, and plugging the anisotropic dispersion $\epsilon_{p\mathbf{k}}$ for the free pairs, we approximate $T_c = 0.218\epsilon_0 \times 2m/(m_p^{xx}m_p^{yy}m_p^{zz})^{1/3}$ in the $g_p \to 0$ limit [25]. Thus, while T_c/ϵ_0 saturates to 0.0629 for a Weyl SOC and to 0.1248 for a YZ SOC in the $1/(m\alpha a_s) \to -\infty$ limit, it reduces to the usual result 0.218 in the $1/(m\alpha a_s) \to +\infty$ limit.

V. SUMMARY

Here we analyzed the properties of BCS-type paired state in a spin-$\frac{1}{2}$ Bose gas with arbitrary SOC. Relying on the mean-field and Ginzburg-Landau theories for the paired state, we showed how the **k**-space structure of a SOC field manifests in the many-body and two-body problems, boosting the stability of the paired state as a function of its strength. For this purpose, we calculated the critical transition temperature for the formation of pairs, and that of the gapless quasiparticle excitations for a broad range of interaction and SOC strengths. It turns out that while the SOC has a minor role in the strong-interaction limit where the ground state at $T = 0$ is a paired BEC, increasing its strength in the weak-interaction limit may allow for the creation of a paired state at much lower temperatures. We also provided a full account of the two-body problem including its nonvanishing binding energy for arbitrarily weak interactions and the anisotropic effective mass tensor.

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[23] The role played by the Hartree shift V_H can also be seen as follows. Due to its analytical simplicity, let us set the SOC to zero, and consider the strong-coupling limit of Eqs. (2) and (3). Since $|\mu| \to \epsilon_b/2 \gg |\Delta_0|$, we find that $N \approx \sum_{\mathbf{k}} |\Delta_0|^2/(2\xi_{\mathbf{k}}^2)$ leading to $|\Delta_0|^2 = [16/(3\pi)]\sqrt{\epsilon_0^3|\mu|}$, and $1/g \approx \sum_{\mathbf{k}}[1/(2\xi_{\mathbf{k}}) + |\Delta_0|^2/(4\xi_{\mathbf{k}}^3)]$ leading to $1/(k_0 a_s) = \sqrt{|\mu|/\epsilon_0} - |\Delta_0|^2/(16\sqrt{\epsilon_0|\mu|^3})$. When we take V_H into account and identify $\mu_p = 2\mu + \epsilon_b = 2V_H - 4\epsilon_0 k_0 a_s/(3\pi)$ as the chemical potential of the pairs, and then match it with $g_p N_p$ of weakly interacting pairs where $g_p = 4\pi a_p/(m_p V)$, $N_p = N_0/2$, and $m_p = 2m$, we find that the effective pair-pair scattering length is given by $a_p = 2mV(U - g/2)/\pi - 2a_s$. Thus, for U that is sufficiently larger than $g/2$, we find that a_p is positive and the system is not prone to collapse. In the absence of a Hartree shift, we note that $a_p = -2a_s < 0$ is in agreement with that of the spinless case [10], and is in sharp contrast with its fermionic counterpart $a_p = 2a_s > 0$ [24].

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