Evolution from BCS to Berezinskii-Kosterlitz-Thouless Superfluidity in One-Dimensional Optical Lattices

M. Iskin[1] and C. A. R. Sá de Melo[2]

[1]*Joint Quantum Institute, NIST and University of Maryland, Gaithersburg, Maryland 20899-8423, USA*
[2]*School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332-0432, USA*

(Received 10 March 2008; revised manuscript received 15 September 2009; published 12 October 2009)

We analyze the finite temperature phase diagram of fermion mixtures in one-dimensional optical lattices as a function of interaction strength. At low temperatures, the system evolves from an anisotropic three-dimensional Bardeen-Cooper-Schrieffer (BCS) superfluid to an effectively two-dimensional Berezinskii-Kosterlitz-Thouless (BKT) superfluid as the interaction strength increases. We calculate the critical temperature as a function of interaction strength, and identify the region where the dimensional crossover occurs for a specified optical lattice potential. Finally, we show that the dominant vortex excitations near the critical temperature evolve from multiplane elliptical vortex loops in the three-dimensional regime to planar vortex-antivortex pairs in the two-dimensional regime, and we propose a detection scheme for these excitations.

DOI: 10.1103/PhysRevLett.103.165301 PACS numbers: 67.85.−d, 03.75.Hh, 03.75.Ss, 05.30.Fk

Ultracold atoms in optical lattices are ideal systems to simulate and study novel and exotic condensed matter phases. Remarkable success has been achieved experimentally with bosonic atoms loaded into three-dimensional (3D) optical lattices, where superfluid and Mott-insulator phases have been observed [1]. In addition, experimental evidence for superfluid and possibly insulating phases were found for fermionic atoms (^{6}Li) in 3D optical lattices [2]. Compared with the purely homogeneous or trapped systems, optical lattices offer additional flexibilities and an unprecedented degree of control such that their physical properties can be studied as a function of onsite atom-atom interactions, tunneling amplitudes between adjacent sites, atom filling fractions and lattice dimensionality. For instance, in strictly two-dimensional (2D) systems the superfluid transition for bosons and fermions is of the BKT type [3,4]. This phase is characterized by the existence of bound vortex-antivortex pairs below the critical temperature $T_{\rm BKT}$, and evidence for it was recently reported in nearly 2D Bose gases confined to one-dimensional (1D) optical lattices [5]. Thus, it is very likely that one of the next research frontiers for experiments with fermions in optical lattices is also the investigation of such a transition.

For bosons or fermions, it is possible to study not only 3D and 2D superfluids as two separate limits, but also the entire evolution from 3D to 2D by tuning the tunneling amplitudes [6,7]. However, fermions offer the additional advantage that their interactions can also be tuned using Feshbach resonances without having to worry about the collapse of the condensate, as it is the case for bosons. Furthermore, the phase diagram of fermions in optical lattices also shows superfluid-to-insulator transitions [8–10] like bosons do. Anticipating experiments, here we study the dimensional crossover from an anisotropic-3D BCS superfluid to an effectively 2D BKT superfluid as a function of interaction strength. We show that vortex excitations near the critical temperature change from elliptical multiplane vortex loops in the anisotropic-3D BCS regime to planar vortex-antivortex pairs in the 2D BKT regime. Finally, we propose an experiment for the detection of vortex excitations.

To describe fermion mixtures in 1D optical lattices, we start with the Hamiltonian ($\hbar = k_B = 1$)

$$H = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} a^\dagger_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma} - g \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \Gamma^*_{\mathbf{k}} \Gamma_{\mathbf{k}'} b^\dagger_{\mathbf{k},\mathbf{q}} b_{\mathbf{k}',\mathbf{q}}, \qquad (1)$$

where the operator $a^\dagger_{\mathbf{k},\sigma}$ creates a fermion with pseudospin σ which labels the hyperfine state of atoms. The operator $b^\dagger_{\mathbf{k},\mathbf{q}} = a^\dagger_{\mathbf{k}+\mathbf{q}/2,\uparrow} a^\dagger_{-\mathbf{k}+\mathbf{q}/2,\downarrow}$ creates fermion pairs with center-of-mass momentum $\mathbf{q}$ and relative momentum $2\mathbf{k}$, while $g > 0$ and $\Gamma_{\mathbf{k}}$ are the strength and symmetry of the attractive interaction between fermions, respectively. Here, $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ with the kinetic energy $\epsilon_{\mathbf{k}} = k_\perp^2/(2m) + 2t_z[1 - \cos(k_z d_z)]$ and the chemical potential μ, where d_z is the lattice spacing along the $\mathbf{z}$ direction.

The saddle-point action for this Hamiltonian is $S_0(\Delta_0^*, \Delta_0) = \beta|\Delta_0|^2/g + (1/M)\sum_{\mathbf{k}}\{\beta(\xi_{\mathbf{k}} - E_{\mathbf{k}}) + 2\ln[(1 + \mathcal{X}_{\mathbf{k}})/2]\}$, where $\beta = 1/T$ is the inverse temperature, M is the number of lattice sites along $\hat{\mathbf{z}}$, $E_{\mathbf{k}} = (\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2)^{1/2}$ is the quasiparticle energy, $\mathcal{X}_{\mathbf{k}} = \tanh(\beta E_{\mathbf{k}}/2)$, and $\Delta_{\mathbf{k}} = \Delta_0 \Gamma_{\mathbf{k}}$ is the saddle-point order parameter. The stationary condition $\partial S_0/\partial \Delta_0^* = 0$ leads to

$$1/g = (1/M)\sum_{\mathbf{k}} |\Gamma_{\mathbf{k}}|^2 \mathcal{X}_{\mathbf{k}}/(2E_{\mathbf{k}}). \qquad (2)$$

We may eliminate g in favor of the binding energy $\epsilon_b < 0$ of two fermions in the lattice potential via $1/g = (1/M)\sum_{\mathbf{k}} |\Gamma_{\mathbf{k}}|^2/(2\epsilon_{\mathbf{k}} - \epsilon_b)$. For s-wave interactions with range $R_0 \sim k_0^{-1}$, we take $\Gamma_{\mathbf{k}} = 1$ for $k < k_0$ and zero otherwise, leading to

$$\epsilon_b = 4t_z - (2t_z^2/\epsilon_0)\exp(1/G) - 2\epsilon_0 \exp(-1/G), \quad (3)$$

where $G = mAg/(4\pi)$ is the dimensionless interaction strength, A is the area in the (x, y) plane and $\epsilon_0 = k_0^2/(2m)$. Notice that two-body bound states in vacuum only exist beyond a critical interaction strength $G_c = 1/\ln(\epsilon_0/t_z)$ for finite t_z, while they always exist for arbitrarily small G in the 2D limit where $t_z \to 0$.

Equation (2) has to be solved self-consistently with the number equation $N_0 = -\partial S_0/(\beta\partial\mu)$, leading to

$$N_0 = \sum_{\mathbf{k}}(1 - \xi_{\mathbf{k}} \mathcal{X}_{\mathbf{k}}/E_{\mathbf{k}}). \quad (4)$$

Solutions to Eqs. (2) and (4) constitute an approximate description of the system only when amplitude and phase fluctuations of the order parameter are small, which is the case only at low temperatures, although quantum fluctuations play a role. However, fluctuations are extremely important close to the critical temperature T_c.

The derivation of the fluctuation action is accomplished by writing the order parameter as $\Phi(q) = |\Delta_0|\delta_{q,0} + \lambda(q)$ with $\lambda(q) = |\lambda(q)|e^{i\theta(q)}$, where $|\lambda(q)|$ is the amplitude and $\theta(q)$ is the phase of the fluctuations. Near T_c, $|\Delta_0|$ vanishes, and the fluctuation action reduces to $S_{\rm fl}(\lambda^*, \lambda) = \beta\sum_q \lambda^*(q)L^{-1}(q)\lambda(q) + (\beta b/2)\sum_q|\lambda(q)|^4$, where the quadratic term is

$$L^{-1}(q) = \frac{1}{g} - \frac{1}{2M}\sum_{\mathbf{k}}\frac{X_{\mathbf{q}/2+\mathbf{k}} + X_{\mathbf{q}/2-\mathbf{k}}}{\xi_{\mathbf{q}/2+\mathbf{k}} + \xi_{\mathbf{q}/2-\mathbf{k}} - iv_\ell}|\Gamma_{\mathbf{k}}|^2, \quad (5)$$

and the quartic term is $b = \sum_{\mathbf{k}}[X_{\mathbf{k}}/(4\xi_{\mathbf{k}}^3) - \beta Y_{\mathbf{k}}/(8\xi_{\mathbf{k}}^2)]$ with $X_{\mathbf{k}} = \tanh(\beta\xi_{\mathbf{k}}/2)$ and $Y_{\mathbf{k}} = \mathrm{sech}^2(\beta\xi_{\mathbf{k}}/2)$.

The analytic continuation $iv_\ell \to \omega + i\delta$ where $\delta \to 0$, and a long wavelength and low frequency expansion leads to $L^{-1}(q) = a + \sum_{ij} q_i c_{ij} q_j - d\omega$. The momentum and frequency independent coefficient is $a = 1/g - (1/M)\sum_{\mathbf{k}} X_{\mathbf{k}}|\Gamma_{\mathbf{k}}|^2/(2\xi_{\mathbf{k}})$; the tensor for low momentum $c_{ij} = \sum_{\mathbf{k}}[\beta^2\dot{\xi}_{\mathbf{k}}^i\dot{\xi}_{\mathbf{k}}^j X_{\mathbf{k}} Y_{\mathbf{k}}/(8\xi_{\mathbf{k}}) + X_{\mathbf{k}}\ddot{\xi}_{\mathbf{k}}^{ij}/(4\xi_{\mathbf{k}}^2)]|\Gamma_{\mathbf{k}}|^2$ is diagonal for s-wave symmetry having the form $c_{ij} = c_i\delta_{ij}$ and $c_\perp \equiv c_x = c_y \neq c_z$. Here, $\dot{\xi}_{\mathbf{k}}^i = \partial\xi_{\mathbf{k}}/\partial k_i$, $\ddot{\xi}_{\mathbf{k}}^{ij} = \partial^2\xi_{\mathbf{k}}/(\partial k_i \partial k_j)$, and δ_{ij} is the Kronecker delta. Finally, the coefficient for low frequency is $d = \lim_{\omega\to 0}\sum_{\mathbf{k}} X_{\mathbf{k}}[1/(4\xi_{\mathbf{k}}^2) + i\pi\delta(2\xi_{\mathbf{k}} - \omega)/\omega]|\Gamma_{\mathbf{k}}|^2$, leading to $L^{-1}(q) = a + c_\perp q_\perp^2 + c_z q_z^2 - d\omega$.

A natural next step is to attempt a phase-only description of the transition to the superfluid state for arbitrary interaction G. However, it is imperative to establish first the region of validity for such phase-only description. For this purpose, we investigate next the importance of amplitude and phase fluctuations of the order parameter as a function of G. So let us consider initially only amplitude fluctuations by ignoring phase fluctuations in the strong attraction regime ($G \gg 1$) corresponding to $\mu < 0$ and $|\mu| \sim |\epsilon_b|/2 \gg t_z$, where $L^{-1}(q) = mA/(4\pi|\epsilon_b|) \times [iv_\ell - \omega_B(\mathbf{q}) + 2\mu_B]$ to lowest order of $\mathbf{q}$ and v_ℓ, with $\omega_B(\mathbf{q}) = q_\perp^2/(2m_{B,\perp}) + 2t_{B,z}[1 - \cos(q_z d_z)]$. After the rescaling $\Psi(q) = \sqrt{mA/(4\pi|\epsilon_b|)}\lambda(q)$, the quadratic term of $S_{\rm fl}$ describes noninteracting bosons with dispersion $\omega_B(\mathbf{q})$, mass $m_{B,\perp} = 2m$ in the (x, y) plane, tunneling amplitude $t_{B,z} = 2t_z^2/|\epsilon_b|$ along the $\hat{\mathbf{z}}$, and chemical potential $\mu_B = 2\mu - \epsilon_b$. Since the quartic term of $S_{\rm fl}$ is small, the resulting Bose gas is weakly repulsive ($g_{B,\perp} = 4\pi/m$ is a constant within our theory), leading to a dominant contribution to the number equation

$$N_{\rm fl} = 2\sum_{\mathbf{q}} n_B[\omega_B(\mathbf{q}) - \tilde{\mu}_B]. \quad (6)$$

Here, $n_B(x) = 1/(e^{\beta x} - 1)$ is the Bose distribution and $\tilde{\mu}_B = \mu_B - V_H < 0$ includes the Hartree shift V_H.

For $G \gg 1$, Eq. (6) leads to Bose-Einstein condensation of tightly bound fermion pairs at $\mathbf{q} = \mathbf{0}$ with $T_c = 2\epsilon_{\rm 2D}/\ln(T_c/t_{B,z})$, where $\epsilon_{\rm 2D} = k_{\rm 2D}^2/(2m)$ is a characteristic energy of fermions in 2D. Here, $k_{\rm 2D}$ is a 2D momentum defined through the 2D density $n_{\rm 2D} = k_{\rm 2D}^2/(2\pi)$. We also define an effective 3D density $n_{\rm 3D} = n_{\rm 2D}/d_z$, where $n_{\rm 3D} = k_{\rm 3D}^3/(3\pi^2)$, and $k_{\rm 3D}$ is the 3D momentum. Notice that, $\epsilon_{\rm 2D} = 2k_{\rm 3D}d_z\epsilon_{\rm 3D}/(3\pi)$, where $\epsilon_{\rm 3D} = k_{\rm 3D}^2/(2m)$ is a characteristic energy in 3D.

For fixed t_z, Eq. (6) shows that T_c is a decreasing function of G. This is most easily seen for a dilute system where $2t_{B,z}[1 - \cos(q_z d_z)] \approx q_z^2/(2m_{B,z})$, such that $m_{B,z} = 1/(2t_{B,z}d_z^2)$ is the effective mass along the $\hat{\mathbf{z}}$. In this case, Eq. (6) gives $T_c \approx 0.218(2m/m_{B,z})^{1/3}\epsilon_{\rm 3D}$, which reduces to the 3D continuum result $T_c = 0.218\epsilon_F$ [11] when $m_{B,z} = 2m$ and $\epsilon_{\rm 3D} \equiv \epsilon_F$. However, $T_c \to 0$ asymptotically when $t_{B,z} \to 0$ or $m_{B,z} \to \infty$, which occurs when the binding energy becomes very large ($|\epsilon_b| \gg t_z$). This limit is clearly unphysical and shows that amplitude fluctuations alone (from the Gaussian theory) cannot recover the BKT transition of tightly bound fermion pairs in the 2D limit, as can be seen in Fig. 1.

To recover the BKT physics in the $G \gg 1$ limit where the paired fermions live in 2D planes, we return to the derivation of the fluctuation action $S_{\rm fl}$ with $t_z = 0$, and include the effects of phase fluctuations as well. Taking the order parameter as $\Phi(x) = [|\Delta_0| + |\eta(x)|]e^{i\theta(x)}$, where $|\eta(x)|$ corresponds to the amplitude fluctuations and $\theta(x)$ is the phase of the order parameter such that $|\Delta_0| \gg |\eta(x)|$, we obtain the phase-only action $S_{\rm fl}(\theta) = (\beta/2)\sum_q[\kappa_0(T)v_\ell^2 + q_i\rho_{ij}(T)q_j]\theta(q)\theta(-q)$. Here, the coefficient $\kappa_0(T) = (1/4)\sum_{\mathbf{k}}[|\Delta_{\mathbf{k}}|^2\mathcal{X}_{\mathbf{k}}/E_{\mathbf{k}}^3 + \beta\xi_{\mathbf{k}}^2\mathcal{Y}_{\mathbf{k}}/(2E_{\mathbf{k}}^2)]$ is the atomic compressibility where $\mathcal{Y}_{\mathbf{k}} = \mathrm{sech}^2(\beta E_{\mathbf{k}}/2)$, and the phase stiffness $\rho_{ij}(T) = n\delta_{ij}/(4m) - \beta\sum_{\mathbf{k}} k_i k_j \mathcal{Y}_{\mathbf{k}}/(8m^2 A)$ is diagonal for the s-wave symmetry: $\rho_{ij}(T) = \rho_0(T)\delta_{ij}$.

This leads to the BKT transition temperature [3,4]

$$T_{\rm BKT} = \pi\rho_0(T_{\rm BKT})/2, \quad (7)$$

which needs to be solved self-consistently with Eqs. (2) and (4) in order to determine $T_{\rm BKT}$, $|\Delta_0|$ and μ as a function of G. Equations (2), (4), and (7) contain corrections due to amplitude and phase fluctuations. Notice that

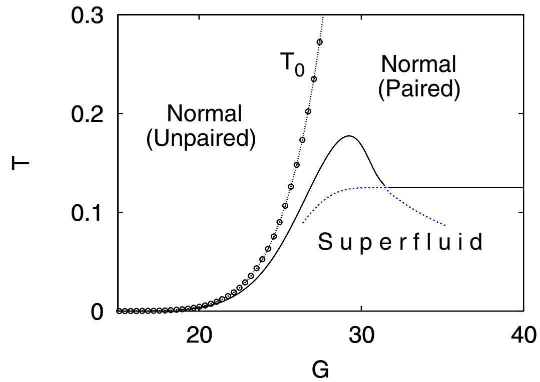

FIG. 1 (color online). Phase diagram of temperature T (in units of ϵ_{2D}) versus $G = mAg/(4\pi)$ with interaction range $k_0 \approx 2 \times 10^4 k_{2D}$ for which $G_c \approx 0.043$, tunneling $t_z \approx 0.043\epsilon_{2D}$, lattice spacing $d_z \approx 0.43\ \mu$m, and planar density $n_{2D} \approx 2.5 \times 10^7\ \text{cm}^{-2}$, such that $k_{2D}d_z \approx 0.55$. T_0 is the saddle-point temperature scale.

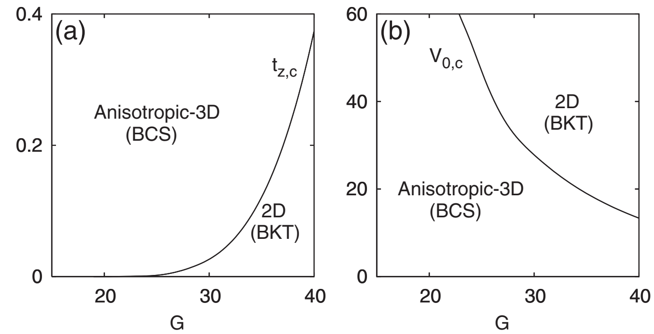

FIG. 2. Characteristic (a) tunneling amplitude $t_{z,c}$ (in units of ϵ_{2D}); and (b) optical lattice depth $V_{0,c}$ (in units of E_r) versus $G = mAg/(4\pi)$ showing the anisotropic-3D to 2D crossover. The parameters are the same as in Fig. 1.

in the weak attraction regime ($G \lesssim 1$), T_{BKT} is identical to the BCS pairing temperature T_0, indicating that fluctuations are not so important for the determination of the critical temperature. Here, T_{BKT} increases with G as $T_{\text{BKT}} = e^{\gamma}\sqrt{2\epsilon_{2D}|\epsilon_b|}/\pi$, where $\gamma \approx 0.577$ is the Euler's constant and $\epsilon_b = -2\epsilon_0 \exp(-1/G)$ is the binding energy in 2D. However, in the strong attraction regime ($G \gg 1$), $T_{\text{BKT}} \ll T_0$ and phase fluctuations completely control the critical temperature. Notice that T_{BKT} saturates to $T_{\text{BKT}} = \epsilon_{2D}/8$ which can be seen in Fig. 1, where $\epsilon_{2D} \equiv \epsilon_F$ is the 2D Fermi energy.

For finite t_z, it can be seen in Fig. 1 that the critical temperature starts deviating substantially from T_0 for $G > 25$, indicating that the phase fluctuation dominated regime is reached. As shown next, it is in this regime that a crossover from anisotropic 3D to 2D behavior occurs. To establish this crossover, we compare the critical temperature $T_{c,\text{Gauss}}$ obtained from the Gaussian theory with the critical temperature T_{BKT} for the BKT transition in the strict 2D limit. When $G \gg 1$, the condition $T_{c,\text{Gauss}} = T_{\text{BKT}}$ leads to $t_{z,c} = \zeta(3/2)\sqrt{\epsilon_{2D}|\epsilon_b|/\pi}/32$, where $\zeta(x)$ is the zeta function.

We can relate t_z to the depth V_0 of the 1D optical latttice potential $V(\mathbf{r}) = V_0 \sin^2(\pi z/d_z)$, where $t_z = (4E_r/\sqrt{\pi})\alpha^{3/4}\exp(-2\sqrt{\alpha})$. Here, $\alpha = V_0/E_r$, where $E_r = \pi^2/(2md_z^2)$ is the recoil energy. Our choice for $t_z \approx 0.043\epsilon_{2D}$ in Fig. 1 corresponds to $\alpha \approx 25$. In Fig. 2, we show the characteristic $t_{z,c}$ and $V_{0,c}$ lines which separate the anisotropic 3D from the 2D regime. When G is fixed, the 2D regime may be reached from the anisotropic 3D regime with increasing V_0 or decreasing t_z. While for fixed V_0 or t_z, the 2D regime may be reached from the anisotropic 3D regime by increasing G.

We can also relate G to the experimentally relevant s-wave scattering length a_s, which in a single-band description is approximately given by [12]

$$\frac{1}{a_s} = \frac{1}{a_c} + \frac{1}{\sqrt{2\pi}\sigma}\ln\left[1 + \frac{|\epsilon_b|}{4t_z} + \sqrt{\frac{|\epsilon_b|}{2t_z} + \frac{\epsilon_b^2}{16t_z^2}}\right], \quad (8)$$

where $a_c = -\sqrt{2\pi}\sigma/\ln[0.915 d_z^2 E_r/(\pi^3\sigma^2 t_z)] < 0$ corresponds to our G_c and it is the minimal scattering length above which a two-body bound state exists, and ϵ_b is given in Eq. (3). Here, $\sigma = d_z \exp(1/4\sqrt{\alpha})/(\pi\alpha^{1/4})$, leading to $d_z/a_c \approx -18$ for $\alpha \approx 25$. Notice that in 3D systems, a bound state exists only for positive values of the scattering length, i.e. $1/a_c \to 0^-$. Using Eq. (8) and the parameters of Fig. 1, we find that the values $G = (0.043, 0.062, 0.1, 1, 40)$ correspond to $d_z/a_s = (-18.48, 0, 16.08, 39.96, 42.56)$. Therefore, the dimensional crossover occurs on the positive side of the Feshbach resonance for these parameters.

So far we showed that the phase-fluctuation dominated regime occurs for $G > 25$, and that a crossover between anisotropic-3D to 2D superfluidity occurs for $G \gg 1$. We may gain further insight into this crossover by rewriting S_{fl} with $t_z \neq 0$ in real space and time such that $S_{\text{fl}}(\lambda^*, \lambda) = (1/V)\int dt dr_\perp dz \mathcal{L}_{\text{fl}}(\lambda^*, \lambda)$, where $\mathcal{L}_{\text{fl}}(\lambda^*, \lambda) = \lambda^*(x) \times (\mathcal{O} - c_z\partial_z^2)\lambda(x) + b|\lambda(x)|^4/2$ is the Lagrangian. Here, V is the volume, $\lambda(x) \equiv \lambda(r_\perp, z, t)$ is the fluctuation field and $\mathcal{O} = a - c_\perp\nabla_\perp^2 - id\partial_t$. Upon discretization $z = nd_z$, S_{fl} reduces to the Lawrence-Doniach (LD) action $S_{\text{LD}}(\lambda_n^*, \lambda_n) = [1/(MA)]\sum_n \int dt dr_\perp \mathcal{L}_{\text{LD}}(\lambda_n^*, \lambda_n)$, where

$$\mathcal{L}_{\text{LD}}(\lambda_n^*, \lambda_n) = \lambda_n^*\mathcal{O}\lambda_n + \frac{c_z}{d_z^2}|\lambda_{n+1} - \lambda_n|^2 + \frac{b}{2}|\lambda_n|^4 \quad (9)$$

is the LD Lagrangian [13]. Here, the local field $\lambda_n \equiv \lambda(r_\perp, z = nd_z, t)$ describes the order parameter in each plane labeled by index n. Writing $a = a_0\epsilon(T)$ with $\epsilon(T) = (T - T_c)/T_c$, scaling the field $\psi_n = \sqrt{b/a_0}\lambda_n$, and defining the correlation lengths $\xi_{0,\perp}^2 = c_\perp/a_0$ and $\xi_{0,z}^2 = c_z/a_0$, and the characteristic time $\tau_0 = -d/a_0$ leads to the scaled action $\tilde{\mathcal{L}}_{\text{LD}}(\psi_n^*, \psi_n) = \psi_n^* i\tau_0\partial_t\psi_n + \epsilon(T)|\psi_n|^2 + \xi_{0,\perp}^2|\nabla_\perp\psi_n|^2 + \xi_{0,z}^2|\psi_{n+1} - \psi_n|^2/d_z^2 + |\psi_n|^4/2$, which describes the system near T_c. Here, $\tilde{\mathcal{L}}_{\text{LD}}(\psi_n^*, \psi_n) = (b/a_0^2)\mathcal{L}_{\text{LD}}(\lambda_n^*, \lambda_n)$. Furthermore, taking $\psi_n = |\psi_n|\exp(i\theta_n)$ in the LD action, such that $|\psi_n| = \phi_0$ is independent of position and time, leads to the phase-only anisotropic-3D XY model with the dimensionless Hamiltonian

$$\tilde{H}_{XY}(r_\perp, n) = K_\perp |d_\perp \nabla_\perp \theta_n|^2 - 2K_z \cos(\theta_{n+1} - \theta_n) + C,$$

where $K_\perp = (\xi_{0,\perp}\phi_0/d_\perp)^2$, $K_z = (\xi_{0,z}\phi_0/d_z)^2$ and $d_\perp \sim k_{2D}^{-1}$, and C is a constant. The dimension-full Hamiltonian is $\tilde{H}_{XY}(r_\perp, n) = (b/a_0^2)H_{XY}(r_\perp, n)$.

It is very important to emphasize that the anisotropic 3D XY model just derived is very different from the standard anisotropic 3D XY model discussed in the context of high-T_c superconductors [14,15]. In that context, the coefficients $K_\perp$ and K_z of the model are studied only for weak attractive interactions (BCS limit) and discussed only as a function of the hopping parameter t_z. In our case, the anisotropic 3D XY model is derived for fixed t_z and investigated as a function of interaction from the weak to strong attraction regimes.

$\tilde{H}_{XY}$ can be mapped onto the vortex-loop representation [15] yielding the dual dimensionless Hamiltonian

$$\tilde{H}_D = \pi \sum_{\mathbf{r}\neq\mathbf{r}'} [K_z J_\perp(\mathbf{r}) \cdot J_\perp(\mathbf{r}') + K_\perp J_z(\mathbf{r}) \cdot J_z(\mathbf{r}')] U(\mathbf{R}),$$

where $U(\mathbf{R} = \mathbf{r} - \mathbf{r}')$ plays the role of an interaction potential for the vortex-loop field $\mathbf{J}(\mathbf{r}) = [J_\perp(\mathbf{r}), J_z(\mathbf{r})]$, and satisfies the differential equation $(\nabla_\perp^2 + \eta^{-2}\partial_z^2)U(\mathbf{R}) = -4\pi\delta(\mathbf{R})$. Here, $\eta = \sqrt{K_\perp/K_z}$ is the anisotropy ratio and $\delta(x)$ is the delta function. The dual transformation maps closed supercurrent flows associated with the gradients of the phase θ onto the vortex-loop vector $\mathbf{J}(\mathbf{r})$, in the same way that the electric current flowing on a ring can be mapped onto a magnetic field vector with the help of the Biot-Savart law.

For large $\mathbf{R} = (R_\perp, R_z)$, the vortex-loop interaction behaves as $U(\mathbf{R}) \sim 1/\sqrt{[R_\perp/(\eta d_\perp)]^2 + (R_z/d_z)^2}$, and leads to equipotentials in the shape of ellipsoids $[R_\perp/(\eta d_\perp)]^2 + (R_z/d_z)^2 = U_0^{-2}$ when $U(\mathbf{R}) = U_0$. Elliptical vortex loops corresponding to a nearly toroidal arrangement of the supercurrent flow are the large scale excitations formed by a continuous closed line having the same potential between segments with $\mathbf{r} = -\mathbf{r}'$. When $\eta \to \infty$, the planes along the $\hat{\mathbf{z}}$ decouple (2D BKT regime) and the vortex loops reduce to planar vortex-antivortex pairs. For $2 < \eta < \infty$, the system is still nearly 2D, and the dominant excitations are square vortex loops coupling two consecutive planes and planar vortex loops. However, in the anisotropic-3D regime when $\eta < 2$, the dominant excitations become multiplane elliptical vortex loops. In the strong attraction regime ($G \gg 1$), $\eta \approx (d_z/d_\perp) \times \sqrt{m_{B,z}/m_{B,\perp}} \approx \sqrt{\epsilon_{2D}|\epsilon_b|}/(4\pi t_z) \gg 1$, and the 2D BKT limit is recovered since $m_{B,z} \gg m_{B,\perp}$. The anisotropic-3D to 2D crossover occurs for $\eta \approx 2$, leading to $t_{z,c} \approx \sqrt{\epsilon_{2D}|\epsilon_b|}/(8\pi)$, which is essentially the same result obtained by equating the Gaussian and BKT critical temperatures.

Since vortex loops are topological objects they preserve their structure in time-of-flight, when the 1D optical lattice along the $\hat{\mathbf{z}}$ is turned off to allow the expansion of the (x, y) pancakes. After sufficient time-of-flight to ensure the overlap of pancakes, vortex loops should appear as dark rings in columnar density images viewed along a direction parallel to the (x, y) plane, since there are no atoms to absorb light in their cores. At temperature T, the ratio of characteristic *in situ* core size of vortex loops in the (x, y) plane $\xi_\perp(T) = \xi_{0,\perp}|\epsilon(T)|^{-2/3}$ and along the $\mathbf{z}$ direction $\xi_z(T) = \xi_{0,z}|\epsilon(T)|^{-2/3}$ is $\xi_\perp(T)/\xi_z(T) \approx 0.91$ for $\eta = 2$ and $k_{2D}d_z = 2.2$. Since typical values of $\xi_{0,\perp} \approx 0.5\ \mu$m, then $\xi_\perp(T) \approx 2.3\ \mu$m and $\xi_z(T) \approx 2.5\ \mu$m at temperatures $T = 0.9T_c$, and vortex loops extend to nearly six planes for an optical lattice with $d_z \approx 0.43\ \mu$m. Smaller values of η or larger values of $k_{2D}d_z$ enlarge $\xi_z(T)$. For parameters $\eta = 1.7$, $k_{2D}d_z = 5.0$ and $T = 0.9T_c$, the ratio $\xi_\perp(T)/\xi_z(T) \approx 0.34$ and $\xi_z(T) \approx 6.8\ \mu$m, such that vortex loops extend to nearly 16 planes in optical lattices with $d_z \approx 0.43\ \mu$m.

We analyzed the finite temperature phase diagram of attractive fermion mixtures in 1D optical lattices. At low temperatures, we found that a dimensional crossover from an anisotropic-3D (BCS) to an effectively 2D (BKT) superfluid occurs as a function of attraction strength even though the tunneling amplitude is fixed. In addition, we discussed that vortex excitations change from elliptical multiplane vortex loops in the anisotropic-3D regime to planar vortex-antivortex pairs in the 2D regime, and suggested an experiment to detect their presence.

We thank the NSF (DMR-0709584) for support.

[1] I. Bloch, Nature Phys. **1**, 23 (2005).
[2] J. K. Chin *et al.*, Nature (London) **443**, 961 (2006).
[3] V. L. Berezinskii, Sov. Phys. JETP **32**, 493 (1971).
[4] J. M. Kosterlitz and D. Thouless, J. Phys. C **5**, L124 (1972).
[5] Z. Hadzibabic *et al.*, Nature (London) **441**, 1118 (2006).
[6] T. Stöferle *et al.*, Phys. Rev. Lett. **96**, 030401 (2006).
[7] S. Ospelkaus *et al.*, Phys. Rev. Lett. **97**, 120403 (2006).
[8] M. Iskin and C. A. R. Sá de Melo, Phys. Rev. Lett. **99**, 080403 (2007).
[9] H. Zhai and T.-L. Ho, Phys. Rev. Lett. **99**, 100402 (2007).
[10] E. G. Moon *et al.*, Phys. Rev. Lett. **99**, 230403 (2007).
[11] C. A. R. Sá de Melo *et al.*, Phys. Rev. Lett. **71**, 3202 (1993).
[12] M. Wouters and G. Orso, Phys. Rev. A **73**, 012707 (2006).
[13] W. E. Lawrence and S. Doniach, in *Proceedings of the 12th International Conference on Low Temperature Physics*, edited by E. Kanda (Keigaku Publ. Co., Tokyo, 1971), p. 361.
[14] P. Minnhagen and P. Olsson, Phys. Rev. B **44**, 4503 (1991).
[15] S. R. Shenoy and B. Chattopadhyay, Phys. Rev. B **51**, 9129 (1995).