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Optimal and Competitive Assortments with Endogenous Pricing Under Hierarchical Consumer Choice Models

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This paper studies assortment planning and pricing for a product category with heterogeneous product types from two brands. We model consumer choice using the nested multinomial logit framework with two different hierarchical structures: a brand-primary model in which consumers choose a brand first, then a product type in the chosen brand, and a type-primary model in which consumers choose a product type first, then a brand within that product type. We consider a centralized regime that finds the optimal solution for the whole category and a decentralized regime that finds a competitive equilibrium between two brands. We find that optimal and competitive assortments and prices have quite distinctive properties across different models. Specifically, with the brand-primary model, both the optimal and the competitive assortments for each brand consist of the most popular product types from the brand. With the type-primary choice model, the optimal and the competitive assortments for each brand may not always consist of the most popular product types of the brand. Instead, the overall assortment in the category consists of a set of most popular product types. The price of a product under the centralized regime can be characterized by a sum of a markup that is constant across all products and brands, its procurement cost, and its marginal operational cost, implying a lower price for more popular products. The markup may be different for each brand and product type under the decentralized regime, implying a higher price for brands with a larger market share. These properties of the assortments and prices can be used as effective guidelines for managers to identify and price the best assortments and to rule out nonoptimal assortments. Our results suggest that to offer the right set of products and prices, category and/or brand managers should create an assortment planning process that is aligned with the hierarchical choice process consumers commonly follow to make purchasing decisions.

Key words: assortment planning; product variety; category management; pricing; inventory costs; nested multinomial logit model
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1. Introduction

Assortment planning is an important decision for retailers and brand manufacturers because product selection that a retailer or a brand offers is one of the main determinants of consumers' store choice and purchasing decisions. Hoping to drive store traffic and increase conversion rates, retailers expanded their assortments dramatically in the past decades and have reached unprecedented levels of variety. Brat et al. (2009) find that a typical grocery store in the United States carries more than 47,000 items, which is more than 50% higher than a decade ago. However, it is not clear if retailers and brand manufacturers necessarily benefit from these assortment expansions because customers may not appreciate such copious levels of product proliferation; more importantly, such a high level of variety leads to a highly fragmented assortment, which in turn results in higher operational costs in the form of more frequent stockouts and higher labor costs. In fact, major retailers such as Walmart, Kroger Co., and Walgreens are starting initiatives to reduce their assortments by 15% (Brat et al. 2009).

Making effective assortment planning decisions requires a sharp understanding of consumer choice processes and a careful balance between the benefits and costs of variety. Both marketing and operations literature provide insights to guide retail assortment planning (see Kök et al. 2008 for a review). The multinomial logit (MNL) consumer choice model

has been widely used in the marketing literature for demand estimation (e.g., Guadagni and Little 1983) and assortment-related issues (Draganska and Jain 2006). Recently, analytical assortment planning models that explicitly consider operational costs have also been developed based on the MNL model (e.g., van Ryzin and Mahajan 1999, Cachon et al. 2005). These MNL-based models assume a homogeneous group of products in a category—possibly variants of the same product, such as dress shirts at a brand store such as Brooks Brothers (single brand, similar features). The applicability of these models may be limited because most retail categories consist of heterogeneous product groups such as men's shirts at a department store (where shirts of different brands and features are presented in the same space) and the products within a subgroup are closer substitutes to each other than are products from another subgroup. The MNL-based models fall short of capturing these interactions in a category with heterogeneous products. This is related to the so-called independence of irrelevant alternatives (IIA) property of the MNL model (Anderson et al. 1992).

In categories consisting of heterogeneous product groups, a nested multinomial logit (NMNL) model as described by Ben-Akiva and Lerman (1985) can be a better alternative to model consumer choice processes. Under the NMNL model does, customers follow a hierarchical choice process, choosing first among subgroups and then a product in the chosen subgroup. The NMNL model provides closed-form choice probabilities much like the MNL model and has been widely used in modeling customer choice processes. Bell and Lattin (1998) model store choice and product choice, and Bucklin and Gupta (1992) and Chintagunta (1993) model purchase incidence and product purchase in nested structures. We are interested in a nested structure within a product category that reflects different subgroups and the heterogeneity in the category.

We consider a product category with multiple product types from two brands. Consumers' choice process follows the NMNL model with one of the following hierarchies. In the brand-primary process, consumers choose first which brand to buy and then a product type within that brand. In the type-primary process, consumers choose first which product type to buy and then one of the brands within that product type. Determining the right hierarchical structure (both the factors involved and their order in the choice process) may be challenging (Urban et al. 1984, Grover and Dillon 1985, Allenby 1989), especially with aggregate consumer data. Kannan and Wright (1991) test alternative nested multinomial logit models for the coffee market and find that both brand-primary and type-primary hierarchical structures fit the data better than the MNL model does, and type-primary structure better explains the data than does the brand-primary. In this paper, we show that the properties of optimal and competitive assortments and prices can be fundamentally different under different hierarchical choice structures (brand-primary versus type-primary). We consider two management regimes: centralized management and decentralized management. Under centralized management, a category manager makes assortment and pricing decisions for the whole category to maximize total category profit.[1] Under decentralized management, such as in a store-within-a-store setting (Jerath and Zhang 2010), two independent brand managers make assortment and pricing decisions for each brand separately to maximize profit of their own brand.[2] In addition to a procurement cost, the retailer incurs the operational costs associated with offering a product in the assortment, such as inventory and replenishment costs, which are concave in the demand volume of the product, reflecting the economies-of-scale effect that is pertinent in retail operations.

Another interpretation of the decentralized models concerns the competition between any two retailers that carry the same category of products. If the retailers' brand equities are strong or their consumers are exclusive (e.g., Costco versus Sam's Club), brand-primary models are more likely to apply (e.g., a customer must decide whether to join Costco or Sam's first, then make a selection within the store/brand chosen). The competition can be also between sellers of the same brand. A consumer considering a Honda Accord may follow a brand-primary or a type-primary choice hierarchy: choose first which dealer to visit and then choose among the product types (colors, options) at the dealer or first choose which type of Accord to buy and then make the location choice by searching dealer inventories. In these settings, the decentralized management represents the state of practice, in which the assortment decisions are made by each retailer and the centralized management represents the efficient system solution.

The contributions of our study based on the above models are threefold. First, we establish that with centralized management under brand-primary model,

[1] It is possible that one of the brands is the category captain, which may lead to possible conflicts of interest (Kurtuluş and Nakkas 2011). The objective of the category captain is usually to meet category performance targets, which include category profits.

[2] A retailer may not allow the brands to independently choose their assortment, but the decentralized regime we consider reflects what the brand manufacturers prefer as their assortment at that particular retailer and may attempt to achieve either by strategic product-line decisions or by offering incentives to the retailer.

the optimal price of each product type is a markup (identical across all products and brands) plus the sum of its procurement cost and its marginal operational cost at its optimal expected demand, and the optimal assortment consists of the most popular product types from each brand. A similar structure for the optimal assortment has been shown by van Ryzin and Mahajan (1999) for a single brand under the MNL model with identical exogenous price across all products. It is known that with exogenously given nonidentical prices, this assortment structure breaks down. The optimal assortment structure under endogenous pricing has not been identified in the literature, even under the MNL model. Our results significantly advance the literature by generalizing the optimal assortment structure to the NMNL model with endogenous pricing for two brands and by demonstrating that this structure based on product popularity is actually valid in more settings than previously known.

Second, we demonstrate that the structures of optimal and equilibrium assortments and prices critically depend on the hierarchical structure of the consumer choice process and the management regime. Under the brand-primary choice model, both the centralized and decentralized assortments consist of the most popular product types from each brand. In contrast, under the type-primary choice model, the assortment structures are more complex, and the assortments do *not* necessarily include only the most popular product types from each brand. Instead, we show that with both management regimes under the type-primary choice model, the product types that are offered by at least one brand consist of a set of the most popular types. Further, under the centralized regime, the product types that are offered by both brands also consist of a set of the most popular types. In other words, the overall assortment in the category consists of a set of most popular product types, and among those product types, more popular types are offered by a larger number of brands. The price for a product type under all management regimes and choice models follows the same simple structure as a sum of a markup, its procurement cost, and its marginal operational cost. We find that the markup is identical across all product types and brands with centralized management under both choice models, identical only for product types offered by the same brand with decentralized management under brand-primary model, and different for all product types with decentralized management under type-primary model. The pricing structure implies that a more popular product would be offered at a lower price (because of decreasing marginal operational costs), and under the decentralized regime, the brand that offers a higher level of variety charges higher prices than does its competition. These results imply that managers must have a clear understanding of the consumer choice process in the category before developing an assortment and pricing strategy. Directly applying the results developed for homogeneous product groups to retail categories with heterogeneous product groups can lead to inferior assortment and pricing decisions.

Third, the properties of the optimal and competitive assortments and prices we show in this paper can greatly simplify the complexity of identifying the right assortments and prices in a category for managers in different retail environments. Without these properties, managers have to rely on full enumeration over all possible assortments whose complexity grows exponentially with the number of products involved. Due to this complexity, managers seek intuitive heuristics to make decisions, such as the "rank-and-select" approach (i.e., rank the products in decreasing order of sales and choose the top k products). However, we find that these heuristics may lead to structurally suboptimal assortments and substantially lower profits. We characterize the properties of optimal assortments under different retailing environments, which can simplify the decision process. Managers only need to consider a limited number of candidate assortments that satisfy these properties and eliminate a large number of non–candidate assortments up front. Our results imply that the commonly used rank-and-select approach needs to be refined to create ranking lists for each subgroup, reflecting the consumers' hierarchical process in the category (for each brand in the brand-primary and for each product type in the type-primary model). Furthermore, each of the candidate assortments would be priced according to the simple pricing structure mentioned above, which allows the expected profit of each of these assortments to be evaluated easily to identify which one offers the highest expected profit for them.

Many decision support models have been developed for assortment planning based on the MNL model and other demand models. Smith and Agrawal (2000) develop an assortment optimization model using a general demand model characterized by the first-choice probabilities and a substitution matrix. Chong et al. (2001) present an empirically based modeling framework using an NMNL model to estimate the revenue and lost sales implication of alternative assortments. Kök and Fisher (2007) describe a methodology for estimation of demand and substitution rates and for assortment optimization using data from a supermarket chain. Cachon et al. (2005) consider assortment planning with endogenous store traffic due to consumer search behavior. Hopp and Xu (2005) study the impact of product modularity on a manufacturer's optimal product line length, and

Bernstein et al. (2011) explore the impact of component commonality on assortment decisions. Grasas et al. (2009) explore the impact of product returns. Aydin and Porteus (2008) optimize inventory levels and prices for multiple products in a given assortment. Maddah and Bish (2007) develop heuristics for joint assortment, inventory, and price optimization. Gaur and Honhon (2006), Honhon et al. (2007), and Alptekinoğlu and Corbett (2008) use a Hotelling-type locational choice model to study assortment planning in a single category. These papers consider categories with homogeneous product groups.

Assortment and pricing decisions have also been studied in competitive markets using the MNL model. Cachon et al. (2008) find that when consumers search for lower prices and better products, lower search costs lead to higher variety, which may support higher prices. Misra (2008) shows how competition and category size influence assortments and prices in an empirically based analysis. Hopp and Xu (2009) study price and assortment competition in a single category and find that competition leads to less variety and lower prices. Dukes et al. (2009) show that strategic assortment reduction may lead to a market with low-variety discount stores and high-variety specialty stores. These papers are based on the MNL model for product categories with homogeneous product groups. Anderson and de Palma (1992) and Cachon and Kök (2007) study price and assortment competition using the NMNL model to capture consumer choice hierarchies on heterogeneous product groups. These papers do not consider choice models with different hierarchical orders.

Our paper is also related to the product line design and pricing literature. Katz (1984) and Moorthy (1984) study product line design problem in monopoly settings. Villas-Boas (1998) and Dong et al. (2009) consider the product line design and pricing problem in distribution channels. Desai (2001), Schmidt-Mohr and Villas-Boas (2008), and Jing and Zhang (2009) extend the problem to competitive settings. In this literature, firms often only offer a limited number of vertically differentiated products to different consumer segments and make decisions on both quality levels and prices of the products without considering operational costs of offering product lines. The optimal prices in these models are often set to ensure that different segments choose different products. In our model, the retailer can offer an arbitrary number of horizontally differentiated products and does take the operational costs of the assortment into account.

The rest of this paper is organized as follows. Section 2 presents the model description. Sections 3 and 4 present our results for brand- and type-primary models. Section 5 provides a comparison across the models and §6 concludes. All proofs are provided in the appendix.

2. Model

We consider retail assortment planning for one product category (e.g., men's shirts; soft drinks). There are different product types in the category (e.g., casual and dress shirts; regular, diet, and caffeine-free soft drinks). Let the set of product types in this category be $T = \{1, 2, \ldots, n\}$. There are two brands (or two retailers), X and Y, which each can offer one product for each product type. Their assortments are denoted $S_X \subset T$ and $S_Y \subset T$, respectively. We consider the case where products in the category are horizontally differentiated with homogeneous quality (e.g., shirts with different colors and soda with different flavors). Thus, we assume that each product in the category has an identical procurement/production cost c per unit. We use subscript t to denote "type" and subscript b to denote "brand" throughout the paper.

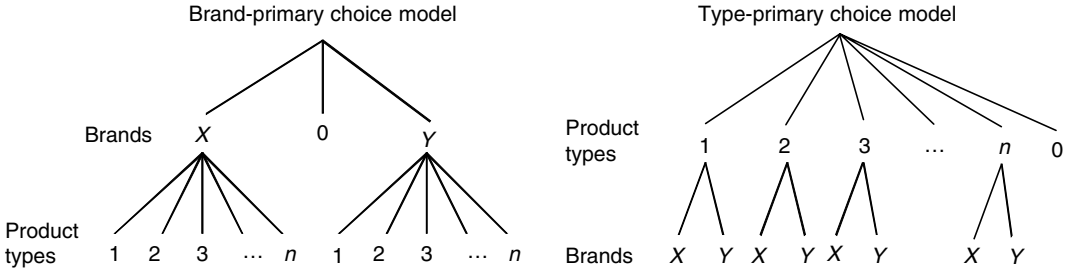
We model consumer choice as a two-stage NMNL process. A consumer's utility U_{bt} associated with purchasing a product type $t \in T$ offered by brand $b \in \{X, Y\}$ is given by

$$U_{bt} = u_{bt} - r_{bt} + \varepsilon_{bt},$$

where u_{bt} is a constant representing the expected utility of the product, which is identical across all consumers; r_{bt} is the selling price of the product; and ε_{bt} is a random variable representing the heterogeneity of the utilities across consumers. Let $\mathbf{r}_b = \{r_{bt},\ t \in S_b\}$ be the price vector for brand $b \in \{X, Y\}$. We assume ε_{bt} are independent and identically distributed zero-mean Gumbel random variables with distribution function $F(x) = \exp[-\exp[-(x/\mu_1 + \gamma)]]$, where $\gamma \approx 0.5722$ is the Euler's constant and μ_1 is the scale parameter. Without loss of generality, we let $\mu_1 = 1$ to simplify exposition. We refer to u_{bt} as the "attractiveness" of the product type t offered by brand b, which can be viewed as a measure of the product's popularity. Therefore, we say a product type k is more popular or attractive than is a product type l offered by a brand b if and only if $u_{bk} > u_{bl}$. We label the product types in the same decreasing order in their attractiveness u_{bt} for brands $b \in \{X, Y\}$, i.e., $u_{b1} \geq u_{b2} \geq \cdots \geq u_{bn}$ for $b \in \{X, Y\}$. We define $v_{bt} = \exp(u_{bt} - r_{bt})$ as the "net attractiveness" of the product type t offered by brand b. In addition to the actual products, consumers have a no-purchase option that is denoted as a faux product 0; that is, a consumer who chooses option 0 does not purchase any product in the category. A consumer's utility associated with the no-purchase option is $U_0 = u_0 + \varepsilon_0$, where u_0 is the expected utility of no-purchase and ε_0 follows the same distribution as ε_{bt}.

Given the assortments and prices $((S_X, \mathbf{r}_X), (S_Y, \mathbf{r}_Y))$, consumers make purchase decisions in a two-stage hierarchical choice process. In this paper,

Figure 1 Brand-Primary and Type-Primary Hierarchical Consumer Choice Models



we consider two different hierarchical choice processes illustrated in Figure 1, both based on the NMNL model. In the brand-primary choice model, a consumer first decides which brand $b \in \{X, Y\}$ or the no-purchase option 0 to choose;[3] if the consumer chooses a brand $b \in \{X, Y\}$ instead of the no-purchase option 0, then she decides which product type $t \in S_b$ offered by brand b to purchase. In contrast, in the type-primary choice model, a consumer first decides which product type $t \in T$ or the no-purchase option 0 to choose; if the consumer chooses a product type $t \in T$ instead of the no-purchase option 0, then she decides which brand that offers the product type t to purchase.

The brand-primary model is suitable to model consumer choice in categories with product types that are not functionally differentiated (e.g., dress shirts with different colors, ice creams with different flavors, detergents with different scents) and strong brand loyalty. In such categories, different product types satisfy more or less the same functional need for consumers, which could make brand differentiation more important. Thus, it is unlikely that consumers would have strong preference for one product type over another, given that the two types mainly provide the same function. Rather, consumers are more likely to develop strong brand preference or loyalty. As a result, they likely choose a brand before they choose a product type. In contrast, the type-primary model is suitable for categories with functionally differentiated product types (e.g., regular drink versus diet drink, regular coffee versus decaf, sedan versus mini-van, point-and-shoot camera versus digital SLR). Consumers choose a product type that satisfies their functional need before selecting a brand. In these categories, such as coffee, consumers may have strong preferences over product types (e.g., regular versus decaf) rather than brands (Kannan and Wright 1991). We recognize that in any category, a mixed choice model, in which some consumers follow the brand-primary model and some others follow the type-primary model, may be a more precise way to describe consumers' behavior. We will discuss later how results for the brand-primary model and the type-primary model actually offer useful insights on the mixed choice model.

We consider two retail assortment management regimes. Under a centralized management regime, the assortments for the category are managed by a single category manager, as is the case for most retailers. The category manager makes centralized decisions on selecting the assortments for both brands (S_X and S_Y) and setting the prices for all products in the assortments ($\mathbf{r}_X$ and $\mathbf{r}_Y$) for the category. The category manager's objective is to maximize the expected profit of the category as a whole. Under a decentralized management regime, for each brand there is one independent brand manager who makes decisions on the assortment and prices for the brand that he or she is responsible for. Each brand manager's objective is to choose the assortment (S_X or S_Y) and the prices for the products in the assortment for the brand ($\mathbf{r}_X$ or $\mathbf{r}_Y$) to maximize the expected profit for his or her brand only. The centralized regime can be also interpreted as a monopoly environment (with two brands), whereas the decentralized regime can be interpreted as a competitive environment.

Let P_{bt} denote the probability that a consumer would choose product type t offered by brand b (we will derive the choice probability P_{bt} later in Equations (3) and (13) for the two choice models, respectively). We normalize the number of customers to one without loss of generality. Therefore, the expected demand of a product type t offered by brand b is just P_{bt}. In addition to the procurement cost c, there is an operational cost associated with offering product type t by brand b, $C(P_{bt})$, which is increasing and concave in the expected demand of the product (i.e., $C'(P_{bt}) \geq 0$, $C''(P_{bt}) \leq 0$). The operational cost will influence the pricing of the products and thereby consumer choices. This operational cost function exhibits economies of scale and is a general representation of the optimized inventory costs that arise in common

[3] Our results can be easily extended to cases where the no-purchase option 0 is available in both stages of the choice hierarchies in both models because the form of the choice probabilities remains the same.

inventory replenishment models such as a newsvendor or a base-stock model for stochastic demand[4] or an economic order quantity model for deterministic demand scenarios[5] (for examples of newsvendor cost function in assortment planning models, see van Ryzin and Mahajan 1999, Gaur and Honhon 2006). In some cases, retailers incur a fixed cost when a product is included in the assortment to reflect the shelf space or administrative costs, which can be modeled with a two-part (fixed plus linear) cost function. All our results hold under this type of cost function as well.

Hence, for given assortments and prices $\{(S_X, \mathbf{r}_X), (S_Y, \mathbf{r}_Y)\}$, the expected profit of product type $t \in S_b$ offered by brand $b \in \{X, Y\}$ can be written as

$$\pi_{bt}((S_X, \mathbf{r}_X), (S_Y, \mathbf{r}_Y)) = (r_{bt} - c)P_{bt} - C(P_{bt}),$$

where the first term is the expected total gross margin of offering product type t by brand b, which is linearly increasing in product type t's expected demand P_{bt}, and the second term is the operational cost for product type t, which is concave and increasing in product type t's expected demand P_{bt}. An optimal assortment has to trade off the benefits of including a product in the assortment (it generates incremental sales and makes its subgroup more attractive) with the costs of including a product in the assortment (it cannibalizes demand for existing products, thereby reducing their revenue and lowering their operational efficiency). Hence, both marketing and operational factorsneed to be considered to make effective assortment decisions.

Generally, the optimal assortment can be found with full enumeration over all possible assortments. In this paper, we aim to characterize some useful structural properties that can be used as guidelines by managers to significantly simplify the process of identifying the optimal and competitive assortments and prices. For some cases, we show that the optimal assortment for each brand is included in the following (small) set, which we call the popular (assortment) set:

$$\mathbf{P} = \big\{\{\}, \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}\big\}.$$

Recall the product variants are labeled in decreasing order of net utility (u_{bt} is decreasing in t for each b). There are only $n+1$ assortments in the set $\mathbf{P}$ for each brand, resulting in $(n+1)^2$ possible optimal assortments for two brands (instead of possible assortments under full enumeration). We use the notation $\mathbf{P}(S)$ to denote the popular set within set S and use the superscript "$*$" to denote optimal or equilibrium decisions under the considered management regime.

3. The Brand-Primary Choice Model

In the brand-primary choice model, a consumer first decides which brand $b \in \{X, Y\}$ or the no-purchase option 0 to choose, then purchases a product type $t \in S_b$ that generates the highest utility to her, i.e., $U_{bt} = \max\{U_{bl}, l \in S_b\}$. Under the NMNL model, the choice probabilities in both stages are specified by the MNL model (see Anderson et al. 1992). According to the MNL model, conditional on that a consumer has chosen a brand $b \in \{X, Y\}$ in the first stage, the probability that she purchases product type t offered in brand b's assortment is given as

$$P(t \mid b) = \frac{v_{bt}}{\sum_{k \in S_b} v_{bk}}. \tag{1}$$

When the consumer makes brand choice in the first stage, her expected utility from choosing a brand b is $U_b = \max\{U_{bk}, k \in S_b\}$, which is the expected utility of the product type that she will purchase in the second stage. As described by Ben-Akiva and Lerman (1985), the Gumbel distribution is closed under maximization, and the maximum utility from choosing a brand b, U_b is also Gumbel distributed with mean

$$E[U_b] = \ln \sum_{k \in S_b} v_{bk}.$$

Thus, according to the MNL model, because

$$\exp\left(\frac{1}{\mu} \ln \sum_{k \in S_b} v_{bk}\right) = \left(\sum_{k \in S_b} v_{bk}\right)^{1/\mu},$$

the probability that the consumer chooses brand b is

$$P(b) = \frac{(\sum_{k \in S_b} v_{bk})^{1/\mu}}{(\sum_{k \in S_X} v_{Xk})^{1/\mu} + (\sum_{k \in S_Y} v_{Yk})^{1/\mu} + v_0}, \tag{2}$$

where $v_0 = \exp(u_0/\mu)$, and μ is the scale parameter that controls the interbrand heterogeneity. Note that the probability of no-purchase is $P(0) = v_0/((\sum_{k \in S_X} v_{Xk})^{1/\mu} + (\sum_{k \in S_Y} v_{Yk})^{1/\mu} + v_0)$. Because we set $\mu_1 = 1$ (which controls intrabrand heterogeneity), if $\mu = \mu_1 = 1$, this NMNL model reduces to the standard MNL model, where all products form a homogeneous set. Therefore, we consider cases with $\mu > 1$, indicating that products across the two brands are less substitutable than are products within a brand (i.e., interbrand heterogeneity is higher than intrabrand heterogeneity). Thus, in the brand-primary model, the probability of a consumer choosing product type t offered by brand $b \in \{X, Y\}$ is given as

$$P_{bt} = P(b)P(t \mid b). \tag{3}$$

[4] In a single-period newsvendor or periodic review base-stock model where (lead time) demand follows a normal distribution with mean d and standard deviation σd^β with $\beta < 1$, optimal stocking level is $s^* = d + z^* \sigma d^\beta$ and the total optimal inventory related costs is given by $(b+h)\phi(z^*)\sigma d^\beta$, where b is the shortage penalty per unit, h is the cost of holding inventory, and $z^* = \Phi^{-1}(b/(b+h))$.

[5] In the economic order quantity (EOQ) model, the retailer incurs an order cost k per order and a holding cost h per unit of inventory held for one period. The well-known EOQ formula suggests ordering every $\sqrt{2k/h\lambda}$ periods, where λ denotes the demand rate. The resulting optimal inventory and order costs is given by $\sqrt{2k\lambda h}$.

3.1. Centralized Management

Under the centralized management regime, the category manager selects the assortments for both brands in the category, S_X and S_Y, and determines the prices for all product types in the assortments selected, $\mathbf{r}_X$ and $\mathbf{r}_Y$, to maximize the expected profit of the category as a whole. Let Π denote the total category profit. The category manager's decision problem can be written as

$$\max_{S_X, S_Y} \max_{\mathbf{r}_X, \mathbf{r}_Y} \Pi((S_X, \mathbf{r}_X), (S_Y, \mathbf{r}_Y)) = \sum_{t \in S_X} \pi_{Xt}((S_X, \mathbf{r}_X), (S_Y, \mathbf{r}_Y)) + \sum_{t \in S_Y} \pi_{Yt}((S_X, \mathbf{r}_X), (S_Y, \mathbf{r}_Y)), \tag{4}$$

where the first term is the expected profit from brand X and the second term is the expected profit from brand Y. We first study the inner maximization problem in (4) to characterize the optimal pricing strategy (for given assortments).

Proposition 1. *Under centralized management with the brand-primary choice model, for any given assortments of the two brands, S_X and S_Y, the optimal price for a product type t offered in the assortment of brand $b \in \{X, Y\}$ is*

$$r_{bt}^* = m^* + c + C'(P_{bt}^*), \tag{5}$$

where

$$m^* = \frac{\mu}{P(0)^*}. \tag{6}$$

The optimal pricing strategy follows a surprisingly simple structure: the optimal price of a product type offered in the assortment of any brand is equal to the sum of a constant category markup m^*, its procurement cost c, and its marginal operational cost at its optimal expected demand $C'(P_{bt}^*)$. The optimal category markup m^* is identical across all product types and brands in the category because m^* given in (6) is independent of both b and t. The identical markup property is unique to the MNL and NMNL models (see Anderson et al. 1992). From (6), we can see that the optimal category markup m^* is increasing in μ and decreasing in $P(0)^*$, which simply indicates that if product types across the two brands are less substitutable and/or the total market share of the retailer is high (possibly because of higher variety), the whole category can afford a higher margin.[6] According to (5), a product type's optimal price increases with its marginal operational cost at its optimal expected demand $C'(P_{bt}^*)$. Because of economies of scale, the marginal operational cost for a product type $C'(P_{bt}^*)$ is decreasing in its optimal expected demand P_{bt}^*. As a result, popular product types (i.e., faster moving products) are priced lower than are niche product types in the category.

To investigate the properties of the optimal assortments in this model, for any given assortment and prices, $(S_X, \mathbf{r}_X)$ and $(S_Y, \mathbf{r}_Y)$, we consider the effect of adding a product type $l \in T \backslash S_X$ with u_{Xl} to brand X's assortment S_X at a given price r_{Xl} on the expected profit of the whole category. Let brand b's new assortment and prices after adding product type l be $S_b^+ = S_b \cup \{l\}$ and $\mathbf{r}_b^+ = (\mathbf{r}_b, r_{bl})$. If $\Pi((S_X^+, \mathbf{r}_X^+), (S_Y, \mathbf{r}_Y)) > \Pi((S_X, \mathbf{r}_X), (S_Y, \mathbf{r}_Y))$, then adding product type l at given price r_{Xl} to S_X would increase the expected profit for the category. Define $\delta = \exp(u_{Xl})$.

Proposition 2. *The expected profit function for the category $\Pi((S_X^+, \mathbf{r}_X^+), (S_Y, \mathbf{r}_Y))$ is a quasiconvex function in δ on the interval $[0, \infty)$. Under centralized management with the brand-primary choice model, the optimal assortment of each brand is always within the popular set $\mathbf{P}$; that is, $S_X^* \in \mathbf{P}$ and $S_Y^* \in \mathbf{P}$ and the prices of products are optimally set following* (5).

As the above proposition indicates, the expected profit function for the category after a new product type l with a given price is added, $\Pi((S_X^+, \mathbf{r}_X^+), (S_Y, \mathbf{r}_Y))$, is quasiconvex in the product's base utility.[7] The quasiconvexity of $\Pi((S_X^+, \mathbf{r}_X^+), (S_Y, \mathbf{r}_Y))$ implies that when selecting a product type to include in an existing assortment, the category manager should follow a simple "nothing-or-most-popular" policy: If candidate product types have relatively small utilities, the disadvantage of adding a popular product type (i.e., stronger cannibalization and reduced operational efficiency) would outweigh its advantages (i.e., higher attractiveness of the category and sales). In this case, no new products should be added to the assortment. However, if there is at least one candidate product type with a high enough utility so that adding it into the existing assortment leads to a profit increase, it is optimal to add the most popular product type with the highest utility.

The above nothing-or-most-popular policy implies that the optimal assortments are *always* within the popular set $\mathbf{P}$ when prices are endogenously optimized (even though resulting optimal prices are not necessarily identical). We use an example with $n = 7$ product types that is shown in Figure 2 to illustrate the intuition of this structure of optimal assortments. Assortment (a) in Figure 2 is potentially an optimal assortment for the example because both brands' assortments include the most popular product types in their brands (note that 64 possible such

[6] Kuksov (2004) and Cachon et al. (2008) also demonstrate in different model settings that broader assortments can allow retailers to mitigate competitive pressure and charge higher prices.

[7] Note that $\Pi((S_X^+, \mathbf{r}_X^+), (S_Y, \mathbf{r}_Y)) = \Pi((S_X, \mathbf{r}_X), (S_Y, \mathbf{r}_Y))$ for $v_{Xl} = 0$. Quasiconvexity implies that the function has at most one local minimum.

Figure 2 Assortments Under the Brand-Primary Choice Model

(a) A potentially optimal (equilibrium) assortment under centralized (decentralized) management

Product types

	1	2	3	4	5	6	7
Brand *X*	*X*	*X*	*X*				
Brand *Y*	*Y*	*Y*	*Y*	*Y*			

(b) A nonoptimal (nonequilibrium) assortment under centralized (decentralized) management

Product types

	1	2	3	4	5	6	7
Brand *X*	*X*	*X*	*X*		*X*		
Brand *Y*	*Y*	*Y*	*Y*	*Y*			

assortments exist). In contrast, assortment (b) in Figure 2 can never be an optimal assortment because brand X's assortment is not a popular set (product type 5 is included, whereas a more popular product type 4 is not included). According to the nothing-or-most-popular policy, assortment (b) can be altered to increase profit by either dropping product type 5 or exchanging product type 5 for product type 4 at the current price of product type 5. After this change, reoptimizing the prices for the altered assortment will further increase the profit. Hence, with continued application of this policy, one can show that the optimal assortment is within the popular set **P**. Interestingly, this property does *not* hold when prices are exogenously given and nonidentical. In the above example, if type 4's price is lower than type 5's price, the above exchange could reduce profits.

A similar structure for the optimal assortment has been established by van Ryzin and Mahajan (1999) for a single brand under the MNL model (i.e., for homogeneous set of products) with identical exogenous price across all products. It is known that with exogenously given nonidentical prices, this assortment structure breaks down, as we have explained in the above example. The optimal assortment structure under endogenous pricing has not been identified in the literature, even under the MNL model. Our results advance the literature by generalizing the optimal assortment structure to the NMNL model with endogenous pricing for multiple brands.

The pricing and assortment structure shown in Propositions 1 and 2 can be useful in identifying the optimal assortments and prices for a category. It is only necessary to consider $(n+1)^2$ candidate assortments within the popular set **P** (instead of the full enumeration over all possible assortments). In other words, the structure provides managers with an effective guideline to eliminate a large number of nonoptimal assortments, such as assortment (b) in Figure 2, up front. Then the optimal assortment can be identified by evaluating the profits of each of the $(n+1)^2$ candidate assortments following the simple pricing strategy shown above.

3.2. Decentralized Management

Under the decentralized management regime, there is an independent brand manager who makes pricing and assortment decisions for each brand to maximize the expected profit for the brand. In this case, the two brand managers are in direct competition with each other because the expected profit of one brand depends on the other brand's assortment and price decisions. For a brand $b \in \{X, Y\}$, the brand manager's decision problem can be written as

$$\max_{S_b} \max_{\mathbf{r}_b} \Pi_b((S_b, \mathbf{r}_b), (S_a, \mathbf{r}_a)) = \sum_{k \in S_b} \pi_{bk}((S_b, \mathbf{r}_b), (S_a, \mathbf{r}_a)), \quad \text{for } a, b \in \{X, Y\} \text{ and } a \neq b. \tag{7}$$

The following proposition characterizes brand b's best response pricing strategy for a given assortment S_b and the other brand's assortment and prices.

Proposition 3. *Under decentralized management with the brand-primary choice model, for any assortment and prices* $(S_a, \mathbf{r}_a)$ *offered by brand* a*, brand* b*'s best response is to price a product type* t *in its own assortment* S_b *as follows*:

$$r_{bt}^* = m_b^* + c + C'(P_{bt}^*), \tag{8}$$

where

$$m_b^* = \frac{\mu}{1 - P(b)^*}. \tag{9}$$

The best response pricing strategy under decentralized management (8) has a similar structure as the optimal pricing strategy under centralized management (5). The only difference is that in decentralized management, the brand markup m_b^* is the same across all product types that are offered within a brand only, because m_b^* given in (9) is a function of b and is independent of t. In centralized management, the competition is between the category as a whole and the no-purchase option, but not within the category among product types and brands. In decentralized management, there is brand competition (brands X and Y, and the no-purchase option) in the first stage in the brand-primary model. After consumers made their choice of brand in the first stage, there is no competition among product types within a brand in the sense that all types are owned by the same brand manager. Therefore, in decentralized management, markups are only identical for product types within a brand. Similar to the centralized case, according to the best response pricing strategy (5), popular product types with high expected demand will be priced lower than are niche product types with low expected demand

in a brand's assortment. As seen in (9), the brand markup m_b^* depends on the same two factors: inter-brand heterogeneity μ, and the market share of the competition $1 - P(b)^*$. Hence, a brand with a higher market share (possibly because of a higher variety level) can afford charging higher prices under the decentralized regime.

The next proposition describes the properties of a brand b's best response assortment to the other brand's assortment and prices by considering how adding a product type $l \in T \backslash S_b$ with attractiveness u_{bl} at a given price r_{bl} into brand b's assortment S_b changes the expected profit for brand b. Let brand b's new assortment and prices after adding product type l be $S_b^+ = S_b \cup \{l\}$ and $\mathbf{r}_b^+ = (\mathbf{r}_b, r_{bl})$. Define $\delta = \exp(u_{bl})$.

Proposition 4. *Under decentralized management in the brand-primary choice model*, (i) *the expected profit function for brand* $b \in \{X, Y\}$, $\Pi_b((S_b^+, \mathbf{r}_b^+), (S_a, \mathbf{r}_a))$ *is quasiconvex in* δ *on the interval* $[0, \infty)$; (ii) *brand b's best response to any assortment and prices offered by brand a is always within the popular set* $\mathbf{P}$, *that is,* $S_b^*(S_a, \mathbf{r}_a) \in \mathbf{P}$, *for* $a, b \in \{X, Y\}$ *and* $a \neq b$; *and* (iii) *at any Nash equilibrium* $\{(S_X^*, \mathbf{r}_X^*), (S_Y^*, \mathbf{r}_Y^*)\}$, *the assortments offered by both brands are always within the popular set* $\mathbf{P}$, *that is,* $S_X^* \in \mathbf{P}$ *and* $S_Y^* \in \mathbf{P}$, *and the prices for both brands,* $\mathbf{r}_X^*$ *and* $\mathbf{r}_Y^*$ *are set according to* (8).

In the decentralized price and assortment game under the brand-primary choice model, given the other brand's assortment and prices, the expected profit for a brand when adding a new product type with a given price into any existing assortment is quasiconvex in the newly added product's attractiveness. As in the centralized case, this property implies that a brand should always respond with an assortment containing its own most popular product types and then optimize the prices of these product types. Thus, at any Nash equilibrium both brands offer assortments that contain their most popular product types and price their product types accordingly. This result extends the pricing and assortment structures we obtained in the centralized case to a competitive setting. Given the structure shown in the above proposition, each brand's strategy space is restricted to $n+1$ possible popular assortments in set $\mathbf{P}$ with prices determined by (8). This significantly reduces the complexity of identifying the equilibrium assortments and prices. Figure 2 shows an example with $n = 7$ product types. Assortment (a) in Figure 2 is potentially an equilibrium assortment for the example because both brands' assortments include the most popular product types in the brands. However, assortment (b) in Figure 2 cannot be an equilibrium assortment because brand X's assortment cannot be a best response.

In this section, we have assumed for expositional simplicity that product types for the two brands have the same order of popularity. It is possible to relax that assumption and show that each brand still carries its most popular assortment in both management regimes under the brand-primary choice model.

4. The Type-Primary Choice Model

In the type-primary choice model, a consumer first decides which product type $t \in S_X \cup S_Y$ or the no-purchase option 0 to choose, then purchases the brand that generates the highest utility for her $\max\{U_{bt}, b \in \{X, Y\}\}$. For $b \in \{X, Y\}$, we define

$$z_{bt} = \begin{cases} 0 & \text{if } t \notin S_b, \\ v_{bt} & \text{if } t \in S_b. \end{cases} \tag{10}$$

According to the MNL model, conditional on that she has chosen product type $t \in S_X \cup S_Y$ in the first stage, the probability that a consumer purchases brand b within product type t is given by

$$P(b \mid t) = \frac{z_{bt}}{z_{Xt} + z_{Yt}}. \tag{11}$$

When the consumer makes product type choice in the first stage, her expected utility from choosing a product type $t \in (S_X \cup S_Y)$ is $U_t = \max\{U_{bt}, b \in \{X, Y\}\}$. Again, U_t is Gumbel distributed with mean

$$E[U_t] = \ln(z_{Xt} + z_{Yt}).$$

Thus, according to the MNL model, the probability that the consumer chooses product type t in the first stage is

$$P(t) = \frac{(z_{Xt} + z_{Yt})^{1/\mu}}{\sum_{k \in S_X \cup S_Y} (z_{Xk} + z_{Yk})^{1/\mu} + v_0}. \tag{12}$$

Note that the probability of no-purchase is $P(0) = v_0 / (\sum_{k \in S_X \cup S_Y} (z_{Xk} + z_{Yk})^{1/\mu} + v_0)$. Similarly, for the type-primary choice model, we also consider cases with $\mu > 1$, indicating that products across the two product types are less substitutable than products offered by different brands within a product type (i.e., interproduct-type heterogeneity is higher than intraproduct-type heterogeneity). As a result, under the type-primary model, the probability of a consumer choosing product type t offered by brand $b \in \{X, Y\}$ is

$$P_{bt} = P(t)P(b \mid t). \tag{13}$$

We will show that the structure of the optimal assortments and prices under this hierarchical choice model is somewhat more complex than it is under the brand-primary choice model. For tractability, we assume that the two brands are symmetric within each product type, i.e., $u_{Xt} = u_{Yt} = u_t$ for all t.

4.1. Centralized Management

The category manager chooses the assortments for both brands, S_X and S_Y, and the prices for the assortments, $\mathbf{r}_X$ and $\mathbf{r}_Y$, to maximize the expected profit for the whole category, which is given by (4). The following proposition characterizes the optimal pricing strategy for a given assortment for the category manager under the type-primary model.

Proposition 5. *Under centralized management with the type-primary choice model, for any given assortments of the two brands, S_X and S_Y, the optimal price for a product type t offered in the assortment of brand $b \in \{X, Y\}$ is*

$$r_{bt}^* = m^* + c + C'(P_{bt}^*), \tag{14}$$

where

$$m^* = \frac{\mu}{P(0)^*}. \tag{15}$$

We can see that the optimal pricing structure in centralized management under the type-primary model (given by (14) and (15)) has exactly the same structure as the one in centralized management under brand-primary model (see (5) and (6)). It indicates that the optimal pricing structure in centralized management is not sensitive to the kind of hierarchy in the consumer choice process because under both choice models, from a decision-making perspective, the competition is between the category as a whole and the no-purchase option in the first stage but not within a category among product types and brands. Hence, a category manager only cares about the total sales of the whole category (i.e., $1 - P(0)^*$) and interproduct-type heterogeneity μ when determining the margin for each product.

To characterize some properties of the optimal assortments in the type-primary model, we focus on a special cost function $\tilde{C}(P_{bt}) = \{\kappa + \alpha P_{bt}$ if $P_{bt} > 0$, 0 otherwise$\}$, which models fixed costs associated with inclusion of a product in the assortment. This function, too, exhibits economies of scale (from an operational cost perspective, it is better to offer fewer products with a higher demand volume for each product than to offer a larger assortment with smaller demand for each product) but is easier to work with for tractability.

Proposition 6. *Under centralized management in the type-primary choice model with cost function $\tilde{C}(P_{bt})$, in an optimal assortment,* (i) *the number of brands that offer a more popular product type is no fewer than the number of brands that offer a less popular product type;* (ii) *brand X's assortment and brand Y's assortment together cover a set of most popular product types, that is, $S_X^* \cup S_Y^* \in \mathbf{P}$; and* (iii) *the product types that are offered by both brands cover a set of most popular product types, that is, $S_X^* \cap S_Y^* \in \mathbf{P}$.*

Figure 3 Assortments Under the Type-Primary Choice Model

(a) A potentially optimal (equilibrium) assortment under centralized (decentralized) management

Product types

	1	2	3	4	5	6	7
Brand *X*	*X*	*X*		*X*			
Brand *Y*	*Y*	*Y*	*Y*		*Y*		

(b) A nonoptimal (nonequilibrium) assortment under centralized (decentralized) management

Product types

	1	2	3	4	5	6	7
Brand *X*	*X*	*X*		*X*	*X*		
Brand *Y*	*Y*	*Y*	*Y*		*Y*		*Y*

Unlike the optimal pricing structure, the optimal assortment structure under type-primary choice is different from that under brand-primary choice: The optimal assortment for each brand, S_X^* or S_Y^*, in the type-primary choice model is *not* within the popular set $\mathbf{P}$ any more. In other words, from a brand perspective, the optimal assortment for each brand may not necessarily offer the most popular product types of that brand (such as assortment (a) in Figure 3, which is not within the popular set $\mathbf{P}$ but potentially can be an optimal assortment in this example with $n = 7$ product types). However, as the proposition indicates, the optimal assortment has a "popular set" structure from a product-type perspective, in the sense that a more popular product type should be offered by more brands. Popularity result applies at the product-type level precisely because the category manager first decides which product types to cover in the assortment planning process to mitigate no-purchase probability under the type-primary model.

Given this property of the optimal assortment, the category manager's assortment decision can be simply characterized by two numbers (g, h): offer both brands for the g most popular product types, offer only one brand for the next h product types, and do not offer the remaining product types. According to part (i) of Proposition 6, in the example shown in Figure 3, assortment (b) cannot be optimal because product type 5 has two brands, whereas more popular product types 3 and 4 only have one brand each, and product type 7 has one brand but a more popular type 6 does not have any.

Interestingly, because a more popular product type is offered by more brands in the optimal assortment, it implies that the product types that are covered by at least one brand (i.e., $S_X^* \cup S_Y^*$) in the optimal assortment should be within the popular set $\mathbf{P}$, as described by part (ii) of the proposition. We can see that assortment (a) in Figure 3 satisfies this structure ($S_X^* \cup S_Y^* = \{1, 2, 3, 4, 5\} \in \mathbf{P}$), but assortment (b) does not ($S_X^* \cup S_Y^* = \{1, 2, 3, 4, 5, 7\} \notin \mathbf{P}$). Furthermore, part (iii) of

the proposition states that the product types that are offered by both brands (i.e., $S_X^* \cap S_Y^*$) in the optimal assortment should be within the popular set **P**, too. Similarly, assortment (a) in Figure 3 satisfies this structure ($S_X^* \cap S_Y^* = \{1, 2\} \in \mathbf{P}$), but assortment (b) does not ($S_X^* \cap S_Y^* = \{1, 2, 5\} \notin \mathbf{P}$). These properties of the optimal assortment can be used as effective guidelines for managers to rule out nonoptimal assortments such as assortment (b) in Figure 3.

To summarize, under the type-primary choice model with centralized management, the breadth of the optimal assortment (number of product types offered) is characterized by a popular set, and the depth of the assortment (number of brands) in a popular product type is more than the depth of the assortment in a less popular type. This result can be viewed as a form of the popular set structure applying at the product-type level, even though each brand's assortment may lack that structure.

4.2. Decentralized Management

In decentralized management, a brand manager's objective is to maximize the expected profit for the brand by choosing the assortment and prices for that brand, as given in (7). The following proposition characterizes the brand's best response pricing strategy conditional on its assortment decision to the other brand's assortment and prices under the type-primary model.

Proposition 7. *Under decentralized management with the type-primary choice model, for any assortment and prices* $(S_a, \mathbf{r}_a)$ *offered by brand a, brand b's best response is to price a product type t in its own assortment* S_b *as*

$$r_{bt}^* = m_{bt}^* + c + C'(P_{bt}^*), \tag{16}$$

where

$$m_{bt}^* = \frac{\mu + \sum_{k \in S_b} m_{bk}^* P_{bk}^*}{\mu - (\mu - 1) P(b \mid t)^*}. \tag{17}$$

In the best response pricing strategy under the type-primary model, the markup m_{bt}^* is *not* identical across products or brands. (Recall under decentralized management under the brand-primary model, the markup is identical across product types within a brand.) In the case of symmetric products across brands, the numerator is the same for all products offered by brand b. The $P(b \mid t)^*$ term in the denominator is 1/2 for all the products that are in direct competition with the other brand (i.e., $t \in S_a \cap S_b$). Hence, those products offered by brand b would have an identical margin, call this m_b^{compete}. The term $P(b \mid t)^* = 1$ for all products that enjoy a monopoly in their type (i.e., $t \in S_b \backslash S_a$), implying that those products by brand b would also have an identical margin, call this m_b^{monopoly}. It can be seen that (17) implies $m_b^{\text{monopoly}} > m_b^{\text{compete}}$. To summarize, each brand chooses two margins for its products: a constant higher margin for products that have a monopoly in their type and a constant lower margin for product types that are in direct competition with the other brand. Finally, we can see from (17) that, all else being equal, a brand that has a higher total market share would charge higher margins than does its competitor.

The game under the type-primary choice model is very different from the game under the brand-primary choice model. Suppose the prices are given exogenously. Under the type-primary model, because consumers choose a product type before choosing a brand, a brand manager must decide what product types to offer by explicitly taking into account which product types the other brand offers. For a product type that is offered by the other brand, the brand manager must consider whether to offer the same product type to compete with the other brand for consumers who potentially choose this product type. For a product type that is not offered by the other brand, the brand manager must consider whether to offer the product type that allows the brand to capture all consumers who potentially choose this product type as a monopoly. As a result, the assortment game under the type-primary choice model is a combinatorial game with n binary compete/no-compete decisions to form a pure strategy for each brand, resulting in a total of 2^{2n} candidate solutions for equilibria. The analysis of such a combinatorial game is generally difficult: Papadimitriou (2007) states that the existence of a pure-strategy Nash equilibrium is not guaranteed in general, there often could be multiple equilibria, and that these problems are intractable—in the sense that their computational complexity is in a special class of NP-complete problems. The case with endogenous pricing is even more complex as the strategy space for each brand consists of n binary variables for the assortment and n continuous variables for the price.

The following proposition identifies several interesting properties of a brand's best response to assortment and prices offered by the other brand and of the equilibrium assortments and prices for both brands with the special cost function $\tilde{C}(P_{bt}) = \{\kappa + \alpha P_{bt}$ if $P_{bt} > 0$, 0 otherwise$\}$. Define $\bar{S}_b$ as the set of product types that are not offered by brand b, i.e., $\bar{S}_b = T \backslash S_b$ for $b \in \{X, Y\}$.

Proposition 8. *Under decentralized management in the type-primary choice model with cost function* $\tilde{C}(P_{bt})$, (i) *given any assortment and prices* $(S_a, \mathbf{r}_a)$ *offered by brand a, the best response assortment for brand b contains the most popular product types that are not offered in brand a's assortment* S_a, *that is,* $S_b^*(S_a) \backslash S_a \in \mathbf{P}(\bar{S}_a)$, *and the best response pricing follows* (16); (ii) *brand b always prefers to offer a more popular product that is not in* S_a *than a*

less popular product in S_a; and (iii) *at a Nash equilibrium $\{(S_X^*, \mathbf{r}_X^*), (S_Y^*, \mathbf{r}_Y^*)\}$, brand X's assortment and brand Y's assortment together cover the most popular product types, that is, $S_X^* \cup S_Y^* \in \mathbf{P}$, and the prices for both brands, $\mathbf{r}_X^*$ and $\mathbf{r}_Y^*$ are set according to* (16).

Under the type-primary model, given any assortment and prices offered by the other brand, a brand's best response assortment is not necessarily a popular set. When consumers choose product type before brand, it might be better for a brand to offer a less popular product type that the other brand does not offer rather than offering a more popular product type that the other brand also offers. This is because offering the less popular product type that the other brand does not offer allows the brand to be a monopoly to capture all consumers that choose the product type. Part (i) of Proposition 8 suggests that the brand manager should separate the product types into two independent groups: the ones offered by the other brand, S_a, and the ones *not* offered by the other brand, $\bar{S}_a$. Then the manager's assortment decision can simply reduce to selecting what product types in the set S_a and what product types in the set $\bar{S}_a$ to offer. Within the set $\bar{S}_a$, if a product type is selected, then all the product types in $\bar{S}_a$ that are more popular must also be selected.[8] Hence, the brand manager's best response assortment must contain a set of the most popular product types in the set $\bar{S}_a$ i.e., $S_b^*(S_a) \setminus S_a \in \mathbf{P}(\bar{S}_a)$. However, the brand manager's best response assortment as a whole is not necessarily within a popular set over all product types; that is, $S_b^*(S_a)$ itself may not be within $\mathbf{P}$. Part (ii) of the proposition says that a brand prefers having a monopoly at a more popular product type rather than competing directly with the other brand in a less popular product type. For example, if $S_Y = \{2, 3, 5\}$, then $S_X = \{2, 3\}$ cannot be a best response because brand X is better off dropping one of the products and including product type 1.

We can see from part (iii) of the proposition that the structure of equilibrium assortments under the type-primary choice model and decentralized management is very different from the structure of its counterparts with the brand-primary choice model. According to part (i) of the proposition, each brand's best response assortment contains a set of most popular product types among those product types not offered by the other brand. As a result, in equilibrium, the product types that are offered by at least one of the brands consist of a set of the most popular product types. Otherwise, at least one of the brands is not playing the best response strategy specified in part (i) of Proposition 8. Thus, part (iii) of the proposition suggests that at a Nash equilibrium, the union of the assortments offered by the two brands should cover a certain number of most popular product types, i.e., $S_X^* \cup S_Y^* \in \mathbf{P}$. This property of the equilibrium assortment can be used to rule out assortments that are not a Nash equilibrium such as assortment (b) in Figure 3, which violates the property.

Figure 3 provides examples to illustrate these properties of the best response assortment and equilibrium assortments. Although brand Y's assortment is not a popular set $\mathbf{P}$, assortment (a) can potentially be an equilibrium assortment because the two brands' assortments can be best responses to each other. For brand X, $S_X = \{1, 2, 4\}$ and $\bar{S}_X = \{3, 5, 6, 7\}$. Brand Y offers two most popular products in $\bar{S}_X$. For brand Y, $S_Y = \{1, 2, 3, 5\}$ and $\bar{S}_Y = \{4, 6, 7\}$. Brand X also offers the most popular product in $\bar{S}_Y$. In addition, $S_X \cup S_Y = \{1, 2, 3, 4, 5, 6, 7\}$, which is a popular set $\mathbf{P}$. However, assortment (b) in Figure 3 violates these properties and therefore cannot be an equilibrium assortment. For brand X, $S_X = \{1, 2, 4, 5\}$ and $\bar{S}_X = \{3, 6, 7\}$. Brand Y's assortment does not offer the most popular product types in $\bar{S}_X$ because type 7 is included, but type 6 is not. Similarly, brand X's assortment cannot be a best response. Furthermore, $S_X \cup S_Y = \{1, 2, 3, 4, 5, 7\}$, which is not a popular set $\mathbf{P}$ because type 7 is offered, but type 6 is not.

Finally, all the results on the type-primary models hold for the case of identical exogenous prices and the general concave cost function. In the decentralized case under that setting, we can also show the additional property that $S_X^* \cap S_Y^* \in \mathbf{P}$, implying that, at equilibrium, more popular products in the assortment are offered by a larger number of brands.

5. Summary and Comparison of the Models

Table 1 provides a summary of the properties of the equilibrium and optimal assortments and prices under the assortment planning models we have studied. As seen, in a product category consisting of heterogeneous product groups, the properties of the optimal and equilibrium assortments are similar under different management regimes but are quite different across different consumer choice models. The differences between the pricing strategies under the four cases demonstrate that the pricing strategy is critically dependent on both the management regime and the consumer choice hierarchy. Thus, our results suggest

[8] We cannot establish a similar property for those products offered by brand a (set S_a) with endogenous pricing. However, we can establish the property for the set S_a when all product types have identical given prices. In that case, the best response assortment must contain a set of the most popular product types in the set S_a. As a result, the best response assortment would be a combination of the most popular product types in the set S_a and the most popular product types in the set $\bar{S}_a$. That is, $S_b^*(S_a) = S_{b1} \cup S_{b2}$, where $S_{b1} \in \mathbf{P}(S_a)$ and $S_{b2} \in \mathbf{P}(\bar{S}_a)$.

Table 1 Summary of the Properties of Equilibrium and Optimal Assortments and Prices

Brand-primary choice model	Type-primary choice model
Centralized management	
$r^*_{bt} = m^* + c + C'(P^*_{bt})$ Identical m^* for all brands, types $S^*_X \in \mathbf{P}$ and $S^*_Y \in \mathbf{P}$	$r^*_{bt} = m^* + c + C'(P^*_{bt})$ Identical m^* for all brands, types $S^*_X \cup S^*_Y \in \mathbf{P}$ $S^*_X \cap S^*_Y \in \mathbf{P}$
Decentralized management	
$r^*_{bt} = m^*_b + c + C'(P^*_{bt})$ Identical m^*_b for types in a brand $S^*_b(S_a) \in \mathbf{P}$ $S^*_X \in \mathbf{P}$ and $S^*_Y \in \mathbf{P}$	$r^*_{bt} = m^*_{bt} + c + C'(P^*_{bt})$ Nonidentical m^*_{bt} $S^*_X \cup S^*_Y \in \mathbf{P}$

that when selecting their assortments, brand and/or category managers should pay close attention to identify what hierarchical choice process consumers commonly follow to make purchase decisions in the product categories they manage.

The properties summarized in Table 1 also offer useful insight to the assortments and prices under more general mixed or overlapping consumer choice processes where some consumers follow the brand-primary model and others follow the type-primary model. An interesting observation from Table 1 is that for both management regimes, the assortment and price properties under the brand-primary model are stronger than those under the type-primary model (i.e., if $S^*_X \in \mathbf{P}$ and $S^*_Y \in \mathbf{P}$ are satisfied, then so do $S^*_X \cup S^*_Y \in \mathbf{P}$ and $S^*_X \cap S^*_Y \in \mathbf{P}$; nonidentical m^*_{bt} for product types is more general than identical m^*_b). Therefore, under a mixed choice model, the assortment and price properties should be similar to those under the type-primary model (i.e., to the weaker set of conditions). Indeed, for such mixed choice models, we can prove in a special case with identical exogenous prices and the cost function $\tilde{C}(P_{bt})$ that the optimal assortments under centralized management have exactly the same properties as those under the type-primary model (i.e., $S^*_X \cup S^*_Y \in \mathbf{P}$ and $S^*_X \cap S^*_Y \in \mathbf{P}$).

We have conducted a numerical study to investigate the optimal and competitive assortments and their impact on profit and variety levels. The numerical study contains experiments with the following parameters: $C(P) = P^\beta$ with $\beta \in \{0.2, 0.4, 0.6\}$, $u_0 \in \{2.18, 0, 3\}$, $\mu \in \{1.428, 1.1\}$, $u_{Xt} = 12 + e^{-t}$, $u_{Yt} = 12 + e^{-t}$ or $u_{Yt} = 11.9 + e^{-t}$, and $c_t = 0$ for all t. We used the general cost function with both hierarchical models so that we can investigate the impact of inventory costs on the outcome. This leads to a total of 36 instances. The base case example is set by choosing the first parameter value in the above list for each variable. In the case with fixed prices, $n = 7$ and $r_t = 10$ for all t. The case of endogenous pricing is computationally more challenging both in centralized and decentralized cases. Especially in the decentralized case, we do not know if an equilibrium exists or not, and enumerating the 2^{2n} assortments is not sufficient to find an equilibrium of the game. For each combination, we also need to solve the pricing equations simultaneously with the assortment decisions. (Centralized cases do not pose this problem because we can solve the pricing problem sequentially for each assortment combination.) As a result, for the decentralized games with endogenous pricing, we have only been able to find equilibria for cases with $n \le 3$ under the brand-primary model and for $n \le 2$ under the type-primary model.

The results of the numerical study are summarized in Table 2. As seen from the middle part of the

Table 2 Summary of the Numerical Study

	Fixed prices		Endogenous prices		
	Brand-primary	Type-primary	Brand-primary	Type-primary	
Number of products	$n = 7$	$n = 7$	$n = 3$	$n = 2$	$n = 1$
Number of cases with equilibrium (out of 36)	36	36	20	10	36
Impact of competition (decentralized system relative to centralized system)					
Average profit loss (min, max) (%)	14.8 (0, 37)	16.8 (0, 43)	76.6 (56, 92)	79.8 (72, 85)	81.8 (66, 92)
Average increase in variety (min, max) (%)	304.9 (0, 1,300)	187.9 (0, 1,300)	25.0 (−50, 100)	0 (0, 0)	33 (0, 100)
Average decrease in markup (min, max) (%)	N/A	N/A	68.0 (56, 78)	75 (70, 80)	74 (67, 80)
Cost of model misspecification under centralized system					
Average profit loss (min, max) (%)	3.4 (0, 27)	5.1 (0, 39)	1.6 (0, 10)	1.3 (0, 8)	0.9 (0, 6)

Note. N/A, the markups are the same under both management regimes because prices are fixed.

table, competition in the decentralized model leads to lower profits and higher variety levels as compared to the centralized model. It can be seen that the profit loss caused by competition is more pronounced in the case of endogenous pricing because of significantly lower margins in the decentralized case. This suggests that price competition between brands is much more detrimental to the profitability of the brands than assortment competition. In the case of fixed pricing, variety is the only competitive lever for the firms. Hence, competition usually leads to a large increase in variety. The variety increase due to competition is more modest in the endogenous pricing case because the margins are significantly lower. (Variety even decreases under competition in one case because of the significantly lower margins.) The last row describes the impact of model misspecification. We compute the profit that one would achieve by implementing the optimal centralized solution (assortment and prices) of the type-primary model when the underlying model is actually the brand-primary model and vice versa. As seen, the cost of model misspecification can be as high as 39% in the case of fixed prices and 10% in the case of endogenous pricing even in our numerical study with small number of products ($n = 7$ and $n = 3$, respectively). This observation demonstrates again the importance of clearly understanding the consumer choice hierarchy and choosing the right assortment and pricing model accordingly in category management.

The following observations from the numerical study are worth noting. First, the assortments under the type-primary model provide a broader (more types) but a thinner (less number of brands in each type) coverage than under the brand-primary model. Consider the base-case example under centralized management and fixed pricing. In the brand-primary case, $S_x^* = S_y^* = \{1\}$. In the type-primary case, $S_X^* \cup S_Y^* = \{1, 2, 3\}$ and $S_X^* \cap S_Y^* = \varnothing$. This is because in the type-primary model, the brands compete with the no-purchase option and each other at the type level (rather than competing at an aggregate brand level in the brand-primary model) and have to cover more product types. Second, because of the combinatorial nature of the game, there are multiple equilibria in many of the decentralized cases of the type-primary model, even in the case with asymmetric brands. In the type-primary model, with asymmetric brands, the optimal assortment contains the more popular brand (brand X) from each product type, whereas the equilibrium assortments may not. If brand Y occupies a product type, brand X may not offer that product type, which results in a lower profit equilibrium than in the case where X offers it and Y does not.

We also investigate the sensitivity of the solution to the system parameters. An increase in the utility of the no-purchase option results in weakly higher variety because a higher level of variety is needed to compete with a stronger outside option. Increasing parameter β of the cost function results in broader optimal and competitive assortments because a higher value of β implies a less concave cost function (as costs become more linear indicating less economies of scale, it is optimal to offer a larger assortment for each brand). Finally, decreasing μ (which implies less heterogeneity, i.e., the competition between subgroups increases relative to the competition within each subgroup) results in a higher variety for the brand-primary cases (as adding more product types within each brand steals more demand from the other brand and the outside option) and a lower variety in the type-primary cases (as less popular product types are now more vulnerable and lose demand to more popular product types and, therefore, are more likely to be excluded from the assortment).

6. Conclusion

Choosing the right assortment and prices for a product category with heterogeneous product groups is an important and complex decision for retailers, category managers, and brand manufacturers. Our paper explores the properties of the optimal and competitive assortments and prices under two hierarchical consumer choice models. These properties can be used to reduce the complexity of finding and pricing the optimal and equilibrium assortments for managers. We find that the optimal or equilibrium assortments have similar properties under different management regimes within the same consumer choice model but have quite distinctive properties across different consumer choice models. Specifically, under the brand-primary choice model, both the centralized and decentralized assortments for each brand consist of the most popular product types from the brand. This property can assist managers to restrict attention to the $(n + 1)^2$ candidate assortments instead of full enumeration of all possible assortments when identifying the optimal and competitive assortments. Under the type-primary choice model, for both management regimes, each brand's assortment may not always consist of the most popular product types of the brand. Instead, a more popular product type would be offered by a larger number of brands, and the product types that are offered by at least one brand and the product types that are offered by all brands include the most popular product types. Using these properties, finding the optimal or the equilibrium assortments with the type-primary choice model under the centralized regime can be reduced to a two-dimensional optimization over two numbers (g, h): offer both brands from the g most popular product types, offer one brand from the next h product types, and do not offer the remaining product types.

The optimal price for a product type under all models follows a simple structure as a sum of a markup, unit procurement cost, and marginal operational cost. We find that a constant category markup is charged across all products and brands under centralized management and that a brand with a larger market share can afford charging higher prices in the competitive cases. Furthermore, all else being equal, we find that the optimal or equilibrium prices of more popular products are lower than less popular products (because of the economies of scale in operational costs). Our results imply that to offer the right set of products and prices, category and/or brand managers should pay close attention to identify what hierarchical process consumers commonly follow to make purchase decisions in product categories they manage.

In numerical comparisons across the models, we find that competitive assortments (brand competition or competition between different retailers) offer more variety and lower prices than do those under centralized management. Connecting this with the Honda dealers example given earlier, Honda prefers that its dealers in a certain area provide lower variety and engage less in price competition (relative to the decisions independent dealers would make). Our results also suggest that price competition is more detrimental to profitability than is assortment competition.

There are limitations of the models presented in this paper. Our results under type-primary (brand-primary) model would have implications for those categories in which majority of customers follow a type-primary (brand-primary) hierarchy. However, there may exist multiple segments in the customer base, possibly each with a different hierarchy or a different preference order over the product types. A mixed NMNL model (Kannan and Wright 1991) or a mixed MNL model (Allenby and Rossi 1999) may fit the data better, respectively, in those cases. For cases with a mixed hierarchy, type-primary results characterized in this paper are applicable. However, with segments with different preference orders, optimal assortments may not have any structure, and Miranda Brondt et al. (2009) show that the assortment optimization problem is NP-hard even with fixed prices. Finally, in MNL-like choice models, a broader assortment would be more attractive to consumers, thereby generate a higher demand. Marketing researchers have long argued that broader assortments may not necessarily be more desirable for consumers because of factors such as evaluation costs (Villas-Boas 2010, Kuksov and Villas-Boas 2010). It may be interesting to consider the effect of those factors on the decisions under decentralized and centralized management regimes.

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Appendix

Proof of Proposition 1. The first-order condition of (4) with respect to r_{bt} for $t \in S_b$ and $b \in \{X, Y\}$ can be written as

$$P_{bt} + (r_{bt} - c - C'(P_{bt}))\frac{\partial P_{bt}}{\partial r_{bt}} + \sum_{k \in S_b,\, k \neq t} (r_{bk} - c - C'(P_{bk}))\frac{\partial P_{bk}}{\partial r_{bt}} + \sum_{k \in S_a,\, a \neq b} (r_{ak} - c - C'(P_{ak}))\frac{\partial P_{ak}}{\partial r_{bt}} = 0.$$

Plugging in the partial derivatives, and collecting terms, we have

$$r_{bt} - c - C'(P_{bt}) = 1 + \frac{\mu - 1}{\mu} \sum_{k \in S_b} (r_{bk} - c - C'(P_{bk}))P(k \mid b) + \frac{1}{\mu} \sum_{a \in \{X, Y\}} \sum_{k \in S_a} (r_{ak} - c - C'(P_{ak}))P_{ak},$$

which implies that $r_{bt} - c - C'(P_{bt})$ is identical for all $t \in S_b$, i.e., $r_{bt} - c - C'(P_{bt}) = r_{bk} - c - C'(P_{bk}) = m_b$, for all $t, k \in S_b$. Therefore, plugging in m_b and rearranging terms, we have

$$m_b = \mu + \sum_{a \in \{X, Y\}} \sum_{k \in S_a} (r_{ak} - c - C'(P_{ak}))P_{ak}, \tag{18}$$

which implies that m_b is identical for all $b \in \{X, Y\}$. Hence, the optimal pricing for any given assortments, S_X and S_Y is to add an identical category markup on top of each product type's costs. So from (18) we have

$$m^* = \mu + m^* \sum_{a \in \{X, Y\}} \sum_{k \in S_a} P^*_{ak},$$

or

$$m^* = \frac{\mu}{1 - \sum_{a \in \{X, Y\}} \sum_{k \in S_a} P^*_{ak}} = \frac{\mu}{P(0)^*},$$

and the optimal price r^*_{bt} can be written as

$$r^*_{bt} = m^* + c + C'(P^*_{bt}). \quad \square$$

Proof of Proposition 2. The retailer offers assortment $S_X \cup S_Y$ where $S_X \subset T$ and $S_Y \subset T$. Let us consider adding a product l with attractiveness u_{Xl} at a given price r_{Xl} to set S_X. Recall $\delta = \exp(u_{Xl})$. Define $\Theta = \sum_{j \in S_X} v_{Xj} + \delta \exp(-r_{Xl})$, $\Omega = \sum_{j \in S_Y} v_{Yj}$, $\Delta = \Theta^\gamma + \Omega^\gamma + v_0$, and $\gamma = 1/\mu$.

The derivative of total profit with respect to δ after collecting terms is given as

$$\frac{\partial \Pi(S_X \cup \{l\}, S_Y)}{\partial \delta} = \frac{\exp(-r_{Xl})}{\Delta^2 \Theta^{-\gamma+2}} \Bigg((r_{Xl} - c - C'(P_{Xl}))\big((\Omega^\gamma + v_0)\gamma\delta \exp(-r_{Xl}) + \Delta(\Theta - \delta \exp(-r_{Xl}))\big) + \sum_{k \in S_X} (C'(P_{Xk}) - r_{Xk} + c)v_{Xk}(\Delta - (\Omega^\gamma + v_0)\gamma) + \sum_{k \in S_Y} (C'(P_{Yk}) - r_{Yk} + c)v_{Yk}^{\gamma-1}\Omega\gamma\Theta \Bigg).$$

All the terms in the numerator are increasing in δ: P_{Xl} goes up with δ so does $-C'(P_{Xl})$ (therefore, so does $r_{Xl} - c - C'(P_{Xl})$); P_{Xk} goes down, so $C'(P_{Xk})$ increases (so does $C'(P_{Xk}) - r_{Xk} + c$); $(\Omega^\gamma + v_0)\gamma\delta\exp(-r_{Xl}) + \Delta(\Theta - \delta\exp(-r_{Xl}))$ is increasing in δ; $\Delta - (\Omega^\gamma + v_0)\gamma = \Theta^\gamma + (\Omega^\gamma + v_0)(1-\gamma)$ is increasing in δ, too; and P_{Yk} goes down, so $C'(P_{Yk})$ increases (so does $C'(P_{Yk}) - r_{Yk} + c$). The denominator is positive (it is actually increasing in δ). Thus, total profit function can have at most one stationary point, i.e., one local optimum, which implies that the derivative is first negative and then positive. Hence, the total profit function $\Pi(S_X \cup \{l\}, S_Y)$ is quasiconvex in δ.

Let S_b^* be an optimal assortment with cardinality i and optimal prices r_{bt}^* for $b \in \{X, Y\}$ and $t \in S_b \subset T$. Suppose $S_b^* \notin \mathbf{P}$; then there exists a $j \notin S_b^*$ such that $u_{bj} > u_{bi}$. However, from the quasiconvexity of $\Pi(S_X \cup \{l\}, S_Y)$, it must also be true that we can either remove i or exchange it for j at the same price r_{bi}^* without decreasing profits. Redefine S_b^* to be this new assortment, and the profit of the assortment can be further increased by reoptimizing the prices for all products in it. Hence, any optimal assortment satisfies $S_b^* \in \mathbf{P}$. □

Proof of Proposition 3. The first-order condition of (7) with respect to r_{bt} for $t \in S_b$ and $b \in \{X, Y\}$ can be written as

$$P_{bt} + (r_{bt} - c - C'(P_{bt}))\frac{\partial P_{bt}}{\partial r_{bt}} + \sum_{k \in S_b,\, k \neq t} (r_{bk} - c - C'(P_{bk}))\frac{\partial P_{bk}}{\partial r_{bt}} = 0.$$

Plugging the partial derivatives and collecting terms, we have

$$r_{bt} - c - C'(P_{bt}) = 1 + \sum_{k \in S_b} (r_{bk} - c - C'(P_{bk})) \cdot \left(-\frac{1}{\mu}(1 - P(b))P(k \mid b) + P(k \mid b) \right),$$

which implies that $r_{bt} - c - C'(P_{bt})$ is identical for all $t \in S_b$, i.e., $r_{bt} - c - C'(P_{bt}) = r_{bk} - c - C'(P_{bk}) = m_b$, for all $t, k \in S_b$. Therefore, plugging in m_b, we have

$$m_b = 1 + m_b \sum_{k \in S_b} \left(-\frac{1}{\mu}(1 - P(b))P(k \mid b) + P(k \mid b) \right).$$

Rearranging terms, we have

$$m_b^* = \frac{\mu}{1 - P(b)^*},$$

and the optimal price r_{bt}^* can be written as

$$r_{bt}^* = m_b^* + c + C'(P_{bt}^*). \quad \square$$

Proof of Proposition 4. The proof is similar to the proof of Proposition 2. □

Proof of Proposition 5. The proof is similar to the proof of Proposition 1. □

Proof of Proposition 6. Note that all results for the type-primary case apply for the special cost function $\tilde{C}$ for the case of endogenous pricing and for general concave cost functions in the case of fixed identical prices. We use the general cost function where applicable, which provides the necessary information for the proofs of the case of fixed prices.

Recall $u_{Xt} = u_{Yt} = u_t$, for all t, $u_1 > u_2 > \cdots > u_n$, $z_{bt} = v_{bt}$ if $t \in S_b$ for $b \in \{X, Y\}$, and $Z_t = z_{Xt} + z_{Yt}$. Define $S = S_X \cup S_Y$, $\Delta = \sum_{t \in S}(Z_t)^\gamma + v_0$. Rewriting the choice probabilities, we get $P(t) = Z_t^\gamma \Delta^{-1}$ and $P_{Xt} = z_{Xt} Z_t^{\gamma-1} \Delta^{-1}$. The total profit function is given by

$$\Pi((S_X, \mathbf{r}_X), (S_Y, \mathbf{r}_Y)) = \sum_{k \in S} \sum_{b \in \{X, Y\}} [(r_{bk} - c) z_{bk} (Z_k)^{\gamma-1} \Delta^{-1} - C(z_{bk}(Z_k)^{\gamma-1}\Delta^{-1})].$$

We prove the result by showing three properties:

(1) Suppose $l \notin S$. We are going to evaluate the impact of adding a product l with utility u_{Xl} and a given price r_{Xl} to brand X's assortment. Let $\delta = \exp(u_{Xl} - r_{Xl})$. Then $Z_l = \delta$, $\Delta(\delta) = \sum_{k \in S}(Z_k)^\gamma + \delta^\gamma + v_0$, $\eta = \Delta(0)$, and $P_{Xl} = \delta^\gamma / \Delta(\delta)$. Rewriting total profit as a function of δ and differentiating, we get

$$\Pi(\delta) = (r_{Xl} - c)\delta^\gamma \Delta(\delta)^{-1} - C\left(\delta^\gamma \Delta(\delta)^{-1}\right) + \sum_{k \in S} \sum_{b \in \{X, Y\}} \left[(r_{bk} - c) z_{bk} (Z_k)^{\gamma-1} \Delta(\delta)^{-1} - C(z_{bk}(Z_k)^{\gamma-1}\Delta(\delta)^{-1})\right],$$

$$\frac{\partial \Pi(\delta)}{\partial(\delta^\gamma)} = \frac{\gamma\delta^{\gamma-1}}{\Delta(\delta)^2}\left((r_{Xl} - c - C'(\delta^\gamma \Delta(\delta)^{-1}))\eta + \sum_{k \in S} \sum_{b \in \{X, Y\}} \left[-\left((r_{bk} - c) - C'(z_{bk}(Z_k)^{\gamma-1}\Delta(\delta)^{-1})\right) z_{bk}(Z_k)^{\gamma-1} \right] \right).$$

Both terms inside the parenthesis in the first-order derivative are increasing in δ. Hence, the total profit function is quasiconvex in δ. Therefore, among those products that have no presence, one would add either nothing or one brand to the most popular product.

(2) Suppose $l \in S \backslash S_x$. Consider product types i, $j \in S_Y \backslash S_X$, and $u_i > u_j$. According to Proposition 5, all product types in the optimal assortment have identical prices r^* with the special cost function $\tilde{C}(P_{bt}) = \kappa + \alpha P_{bt}$ for any $P_{bt} > 0$. We will show that for given assortments and prices $\{(S_X, \mathbf{r}^*), (S_Y, \mathbf{r}^*)\}$, adding product type i into brand X's assortment results in higher profit than adding product type j. Let's add product type j into X and then reoptimize prices. Let $\mathbf{r}_j^*$ denote the new optimal price vector. The profit of the new assortment can be written as

$$\begin{aligned}\Pi\left((S_X \cup \{j\}, \mathbf{r}_j^*), (S_Y, \mathbf{r}_j^*)\right) &= m_j^* \left(\sum_{S_X \backslash S_Y} \frac{z_{Xt}^\gamma}{\Delta_j} + \sum_{S_X \cap S_Y} \left(\frac{z_{Xt}}{Z_t}\frac{Z_t^\gamma}{\Delta_j} + \frac{z_{Yt}}{Z_t}\frac{Z_t^\gamma}{\Delta_j} \right) \right. \\ &\quad \left. + \sum_{S_Y \backslash S_X} \frac{z_{Yt}^\gamma}{\Delta_j} - \frac{z_{Yj}^\gamma}{\Delta_j} + \frac{z_{Xj}}{Z_j}\frac{Z_j^\gamma}{\Delta_j} + \frac{z_{Yj}}{Z_j}\frac{Z_j^\gamma}{\Delta_j} \right) \\ &= m_j^* \Delta_j^{-1}(\Delta_j - v_0),\end{aligned}$$

where $\Delta_j = \sum_{S_X \backslash S_Y} z_{Xt}^\gamma + \sum_{S_X \cap S_Y} Z_t^\gamma + \sum_{S_Y \backslash S_X} z_{Yt}^\gamma - z_{Yj}^\gamma + Z_j^\gamma + v_0$, and $m_j^* = r_j^* - c - \alpha$.

Now, instead of adding j, if we add product type i into X, but still price the new assortment at the price vector $\mathbf{r}_j^*$, the profit of the new assortment can be written as

$$\Pi\left((S_X \cup \{i\}, \mathbf{r}_j^*), (S_Y, \mathbf{r}_j^*)\right) = m_j^* \Delta_i^{-1}(\Delta_i - v_0),$$

where $\Delta_i = \sum_{S_X \backslash S_Y} z_{Xt}^\gamma + \sum_{S_X \cap S_Y} Z_t^\gamma + \sum_{S_Y \backslash S_X} z_{Yt}^\gamma - z_{Yi}^\gamma + Z_i^\gamma + v_0$.

Because $u_{Xi} = u_{Yi} = u_i$ and $u_{Xj} = u_{Yj} = u_j$, and all product types have identical prices $\mathbf{r}_j^*$, we have $-z_{Yi}^{\gamma} + Z_i^{\gamma} > -z_{Yj}^{\gamma} + Z_j^{\gamma}$, which implies $\Delta_i > \Delta_j$. Because function $f(x) = x^{-1}(x - v_0)$ is increasing in x, we have $\Pi((S_X \cup \{i\}, \mathbf{r}_j^*), (S_Y, \mathbf{r}_j^*)) > \Pi((S_X \cup \{j\}, \mathbf{r}_j^*), (S_Y, \mathbf{r}_j^*))$. Furthermore, after adding i at price vector $\mathbf{r}_j^*$, reoptimizing the prices for the new assortment would lead to even higher profit. Therefore, among those products that have no presence, one would add either nothing or one brand to the most popular product.

(3) Although the first two properties are useful in eliminating suboptimal assortment structures, it does not help eliminate the following case, $S_X = S_Y \notin \mathbf{P}$. An example is $S_X = S_Y = \{2, 3, 5\}$. The above property holds for both S_X and S_Y, but as shown next, this assortment cannot be an optimal solution. Let us consider the impact of adding a product l with utility u_l at a fixed price $r_{Xl} = r_{YL}$ in both brands to an assortment with $S_X = S_Y$. Let $\delta = \exp(u_{Xl} - r_{Xl})$, $Z_l = 2\delta$, and $\Delta(\delta) = \sum_{k \in S}(Z_k)^{\gamma} + (2\delta)^{\gamma} + v_0$. Rewrite the profit as a function

$$\Pi(\delta) = 2[(r_{Xl} - c)2^{\gamma-1}\delta^{\gamma}\Delta(\delta)^{-1} - C(2^{\gamma-1}\delta^{\gamma}\Delta(\delta)^{-1})] + \sum_{k \in S_X} \sum_{b \in \{X,Y\}} [(r_{bk} - c)z_{bk}(Z_k)^{\gamma-1}\Delta(\delta)^{-1} - C(z_{bk}(Z_k)^{\gamma-1}\Delta(\delta)^{-1})].$$

Similar to the above, it can be shown that the profit is quasiconvex in δ, implying that we should add the most popular product or not add anything. Hence, $S_X = \{2, 3, 5\}$ and $S_Y = \{2, 3, 5\}$ cannot be an optimal solution; we can either drop type 5 (in both brands) or exchange it with type 1 at the current price of 5 (in both brands) and improve profit. If 1 is added, reoptimizing prices further improves the profit. These three properties together imply part (i), which in turn implies parts (ii) and (iii) of the proposition. □

Proof of Proposition 7. The first-order condition of (7) with respect to r_{bt} for $t \in S_b$ and $b \in \{X, Y\}$ can be written as

$$P_{bt} + (r_{bt} - c - C'(P_{bt}))\frac{\partial P_{bt}}{\partial r_{bt}} + \sum_{k \in S_b,\, k \neq t} (p_{bk} - c - C'(P_{bk}))\frac{\partial P_{bk}}{\partial r_{bt}} = 0,$$

Plugging the partial derivatives, and collecting terms, we have

$$r_{bt} - c - C'(P_{bt}) = 1 + \frac{\mu - 1}{\mu}(r_{bt} - c - C'(P_{bt}))P(b \mid t) + \frac{1}{\mu}\sum_{k \in S_b}(r_{bk} - c - C'(P_{bk}))P_{bk},$$

which implies that $r_{bt} - c - C'(P_{bt})$ or the margin m_{bt} is not identical for all $t \in S_b$ and $b \in \{X, Y\}$. Rearranging terms, the margin m_{bt} can be written as

$$m_{bt}^* = \frac{\mu + \sum_{k \in S_b} m_{bk}^* P_{bk}^*}{\mu - (\mu - 1)P(b \mid t)^*},$$

and the optimal price r_{bt}^* can be written as

$$r_{bt}^* = m_{bt}^* + c + C'(P_{bt}^*). \quad \square$$

Proof of Proposition 8. We use the definitions that are used in the proof of Proposition 6. To show part (i), we evaluate the total profit of brand X while adding product $l \notin S$ with $\delta = \exp(u_{Xl} - r_{Xl})$ at fixed price r_{Xl} to S_X. Define $\Delta(\delta) = \sum_{k \in S}(Z_k)^{\gamma} + \delta^{\gamma} + v_0$ and $\eta = \Delta(0)$.

$$\Pi_X(\delta) = (r_{Xl} - c)\delta^{\gamma}\Delta(\delta)^{-1} - C(\delta^{\gamma}\Delta(\delta)^{-1}) + \sum_{k \in S_X}[(r_{Xk} - c)z_{Xk}(Z_k)^{\gamma-1}\Delta(\delta)^{-1} - C(z_{Xk}(Z_k)^{\gamma-1}\Delta(\delta)^{-1})],$$

$$\frac{\partial \Pi_X(\delta)}{\partial(\delta^{\gamma})} = \frac{\gamma\delta^{\gamma-1}}{\Delta(\delta)^2}\Bigg((r_{Xl} - c - C'(\delta^{\gamma}\Delta(\delta)^{-1}))\eta + \sum_{k \in S_X}[-((r_{Xk} - c) - C'(z_{Xk}(Z_k)^{\gamma-1}\Delta(\delta)^{-1}))z_{Xk}(Z_k)^{\gamma-1}]\Bigg).$$

Both terms inside the parenthesis in the first-order derivative are increasing in δ. This suggests that the profit function of a brand is quasiconvex in δ. Hence, among those products that are not carried by the other brand, a brand should always add only the most popular product types.

For part (ii), given $(S_Y, \mathbf{r}_Y)$, suppose the best response is $(S_Y, \mathbf{r}_Y)$ such that $S_X = S_Y \notin \mathbf{P}$, e.g., $S_X = S_Y = \{2, 3, 5\}$. We show that brand X's assortment cannot be a best response because exchanging any of the products in S_X with a more popular product outside S_Y, e.g., product type 1, at the same price and then reoptimizing brand X's prices leads to a higher profit. Recall that the margins of all products by X that are also in S_Y have the same margin, call it m_X, as discussed after Proposition 7 because we assume brand X and brand Y products are symmetric within each product type. Also $\tilde{C}(P_{bt}) = \kappa + \alpha P_{bt}$ for any $P_{bt} > 0$. Thus, prices of the products in S_X are also identical: $r_{Xt} = m_X + c + C'(P_{Xt}) = m_X + c + \alpha$ for all $t \in S_X$. Let $r \stackrel{\text{def}}{=} r_{Xt}$. Then $v_t = \exp(u_t - r)$, $\Delta = v_0 + \sum_{k \in S_X}(2v_k)^{\gamma}$, and $P_{Xt} = (1/2)(2v_t)^{\gamma}\Delta^{-1}$. The profit of brand X is given by

$$\Pi_X = \sum_{k \in S_X}[(r_{Xk} - c - \alpha)P_{bk}] - \kappa|S_X| = m_X \sum_{k \in S_X} P_{bk} - \kappa|S_X|.$$

Now we replace any j in S_X with a (weakly) more popular product l (i.e. $u_l \geq u_j$) that is not in $S_X = S_Y$, keeping its price the same as j. Then $v_l = \exp(u_l - r)$, $\theta = v_0 + \sum_{k \in S_X \setminus \{j\}}(2v_k)^{\gamma} + v_j^{\gamma} + v_l^{\gamma} = \Delta + v_j^{\gamma}(1 - 2^{\gamma}) + v_l^{\gamma}$, and $P_{Xt} = (1/2)(2v_t)^{\gamma}\theta^{-1}$ if $t \in S_Y$ and $P_{Xl} = v_l^{\gamma}\theta^{-1}$. Let Π_X^e denote the profit of brand X after the exchange.

$$\Pi_X^e - \Pi_X = m_X\left(v_l^{\gamma}\theta^{-1} + \sum_{k \in S_X \setminus j} \tfrac{1}{2}(2v_t)^{\gamma}\theta^{-1} - \sum_{k \in S_X} \tfrac{1}{2}(2v_t)^{\gamma}\Delta^{-1}\right).$$

Multiplying by $\Delta\theta/m_X$ and defining $A = \sum_{k \in S_X}(1/2)(2v_t)^{\gamma}$, we get

$$\begin{aligned}(A + {v_l}^{\gamma} - 2^{\gamma-1}{v_j}^{\gamma})\Delta - A\theta &= ({v_l}^{\gamma} - 2^{\gamma-1}{v_j}^{\gamma})\Delta - A(v_l^{\gamma} + v_j^{\gamma} - 2^{\gamma}v_j^{\gamma}) \\ &> (2{v_l}^{\gamma} - 2^{\gamma}{v_j}^{\gamma})A - A(v_l^{\gamma} + v_j^{\gamma} - 2^{\gamma}v_j^{\gamma}) > 0,\end{aligned}$$

because $2A < \Delta$ and $v_l \geq v_j$. Hence, exchanging product j with l is more profitable even keeping prices the same. Brand X can optimize prices after this exchange to improve profit further.

Repeated application of the properties of the best-response assortments outlined in parts (i) and (ii) to both brands implies the properties of equilibrium assortments given in part (iii) of this proposition. □

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