

Category Management and Coordination in Retail Assortment Planning in the Presence of Basket Shopping Consumers

Gérard P. Cachon

The Wharton School, University of Pennsylvania, Philadelphia, Pennsylvania 19104,
cachon@wharton.upenn.edu

A. Gürhan Kök

Fuqua School of Business, Duke University, Durham, North Carolina 27708,
gurhan.kok@duke.edu

This paper studies the assortment planning problem with multiple merchandise categories and basket shopping consumers (i.e., consumers who desire to purchase from multiple categories). We present a duopoly model in which retailers choose prices and variety level in each category and consumers make their store choice between retail stores and a no-purchase alternative based on their utilities from each category. The common practice of category management (CM) is an example of a decentralized regime for controlling assortment because each category manager is responsible for maximizing his or her assigned category's profit. Alternatively, a retailer can make category decisions across the store with a centralized regime. We show that CM never finds the optimal solution and provides both less variety and higher prices than optimal. In a numerical study, we demonstrate that profit loss due to CM can be significant. Finally, we propose a decentralized regime that uses basket profits, a new metric, rather than accounting profits. Basket profits are easily evaluated using point-of-sale data, and the proposed method produces near-optimal solutions.

Key words: game theory; assortment planning; optimization

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Assortment planning is both extremely important and challenging for retailers. No assortment planning process is capable of accounting for all of the marketing and operational implications of its decisions due to limited data and the complexity of the task. Organizational forms such as category management (CM) (ACNielsen 1998) and assortment planning models in the literature (e.g., Kök and Fisher 2004) focus on the selection of products in a single category assuming store traffic is exogenous, i.e., prices and variety within a category influence demand conditional on a store visit, but does not influence store choice. For example, many retailers are adopting an “efficient assortment” strategy, which primarily seeks to find the profit maximizing level of variety by eliminating low-selling products (Kurt Salmon Associates 1993). However, if a retailer reduces variety in all categories based on single-category analyses, then the store becomes less attractive and some customers are likely to defect to other retailers, reducing store traffic. This concern is particularly relevant with respect to basket shoppers—consumers who desire to purchase from multiple categories. If a basket shopper does not

find an item that she wants in one category, she may purchase her entire basket from another retailer (Bell and Lattin 1998). Although retailers are well aware of this interdependence across categories, there is little research on what can be done to address it.

This paper develops a stylized model to evaluate CM at two competing retailers in the presence of basket shopping consumers. Each retailer offers two merchandise categories. Retailers determine prices and variety level in each category. Single-category shoppers and basket shoppers choose between the two retailers or a no-purchase option, depending on their utilities in each category at each retailer. A retailer can manage its merchandise categories with a centralized or a decentralized regime. The common practice of CM is an example of a decentralized regime for controlling assortment because each category manager is charged with maximizing profit for his or her assigned category. Because basket shoppers' store choice decision depends on the prices and variety levels of other categories, one category's optimal decisions depends on the decisions of the other categories. Hence, a game-theoretic situation arises. CM

can be interpreted as an explicit noncooperative game between the category managers because each category manager is responsible exclusively for the profits of her own category. Alternatively, it can be interpreted as an iterative application of single-category planning where each category's variety level is optimized assuming all other assortment decisions for the retailer are fixed. Decentralized regimes such as CM are analytically manageable but they ignore (in their pure form) the impact of cross-category interactions. Centralized regimes account for these effects, but actually only in theory because they are not implementable in practice: it is extremely difficult to design a model to account for all cross-category effects, to estimate its parameters with available data, and solve it. Chen et al. (1999) also emphasize the need for models that are solvable with parameters that can be estimated from available data. As CM and centralization are two extremes, merchandising managers often create an intermediate approach by adding constraints to the planning process based on their own knowledge about the store's categories and customers: for example, include a particular product, have at least two brands in a subgroup, and ensure that the timing and depth of promotions are coordinated across obviously complementary products such as chips and salsa or beer and pretzels. It is not clear, however, if the appropriate constraints are implemented (e.g., whether there should be two brands or five brands) or whether all of the necessary interactions are accounted for with this ad-hoc approach.

Previous research shows that CM resulted in a more profitable pricing structure by eliminating the competitive pricing between brands. Zenor (1994) and Basuroy et al. (2001) compare brand management (i.e., decentralized management of competing products) to CM (centralized management of a category) in a single category and find that CM leads to higher prices. Our paper attempts to shift the discussion one level higher by comparing CM with centralized store management. It is expected that decentralization will perform worse than centralization, so the question is whether it performs well under certain conditions. It is also important to assess whether the loss due to decentralization is significant and whether decentralized solutions have a consistent bias (too much or too little variety, too high or too low prices). Finally, is there a way to have the best of both worlds, i.e., are there easily solvable management regimes, based on readily available data, that lead to nearly optimal assortments?

We characterize the assortment chosen in a decentralized regime, which we refer to as CM, as well as the assortment in a centralized solution (OPT). We show that if there are any basket shoppers, CM

provides less variety and higher prices than OPT. CM can lead to poor decisions because the category manager does not sufficiently account for how his or her decisions influence total store traffic. With numerical examples, we demonstrate that the profit loss due to CM can be significant. More importantly, the performance worsens as the number of categories and proportion of basket shoppers increase. These results hold both for a single retailer and in duopoly competition. The dominant strategy for each retailer is to switch to centralized management (OPT). Our point is that decentralization can be costly if there are basket shopping consumers and the interactions among categories is not explicitly modeled. To address the potential problem with a decentralized approach to assortment planning, we propose a simple heuristic that retains decentralized decision making (category managers optimize their own categories' profit) but adjusts how profits are measured. To be specific, instead of using an accounting measure of a category's profit, we define a new measure called *basket profits*. Basket profits can be estimated using point-of-sale data. It enables CM to approximately measure the true marginal benefits of merchandising decisions and lead to near-optimal profits. We believe that this analytical approach is an attractive alternative relative to ad-hoc coordination across category managers.

We review the related literature in the next section. We introduce our model in §2. We present the analysis of the variety competition case (where prices are exogenous) in §3, followed by the price and variety competition case in §4. We present a brief numerical study in §5, discuss alternative demand models in §6, and conclude in §7. Appendix A presents a replenishment system with convex costs. All proofs are presented in Appendix B.

1. Related Literature

Our consumer choice model is built on the random utility approach (see Anderson et al. 1992). Each consumer receives a random utility from each item in the choice set and the highest utility item is chosen. As a result, increasing the breadth and depth of the assortment in our model increases total demand. The findings in Dhar et al. (2001) are generally consistent with that assumption. However, we recognize that our model does not explicitly account for other possible factors that influence the relationship between assortment variety and demand: the space devoted to a category and the presence or absence of a favorite item influence the perception of variety (Kahn and Lehmann 1991, Broniarczyk et al. 1998) as well as the arrangement, complexity, and presence of repeated items in an assortment (Hoch et al. 1999, Huffman and Kahn 1998, Simonson 1999).

Although research has primarily focused on single-category choice decisions, there is recent research that examines multiple category purchases in a single-shopping occasion by modeling the dependency across multicategory items explicitly (see Russell et al. 1997 for a review). Manchanda et al. (1999) find that two categories may co-occur in a consumer basket, either due to their complementary nature (e.g., cake mix and frosting) or due to coincidence (e.g., similar purchase cycles or unobserved factors). Bell and Lattin (1998) show that consumers make their store choice based on the total basket utility. Bodapati and Srinivasan (1999) relates feature advertising to store traffic effects using a nested logit framework. In these papers and in ours, consumers first assign a utility to an anticipated market basket and subsequently use this utility to determine store choice.

Fixed costs for each store visit (e.g., search and travel costs) provide an intuitive explanation for why consumers basket shop. Bell et al. (1998) use market basket data to analyze consumer store choices and explicitly consider the roles of fixed and variable costs of shopping. We do not explicitly derive optimal baskets. We take them as given; however, consumers make optimal store choice given their baskets. Some consumers distribute their shopping across stores to take advantage of discounts and different selections, a behavior known as *cherry-picking*. Fox and Hoch (2005) compare cherry-picking consumers with single-store shoppers and find that consumers with lower income and larger shopping baskets are more likely to engage in cherry-picking.

Price competition across multiple categories has been studied by several researchers using Hotelling type models in which consumers travel and search costs impact retailers' strategies and whether or not consumers cherry-pick. Lal and Rao (1997) show that in equilibrium one firm will adapt every day low pricing, whereas the other firm adapts promotional pricing strategy. The loss-leader literature suggests offering promotions in one category to increase store traffic and overall profit (see, for example, Lal and Matutes 1989). From an empirical perspective, Walters and Mackenzie (1988) report that loss-leader pricing produced only a small increase in store traffic. Recent empirical research in marketing (e.g., Bayus and Putsis 1999, Draganska and Jain 2005) examines the relation of prices with product variety and other marketing mix variables that are endogenously determined by firms and their impact on market shares.

Chen et al. (1999) also study the impact of basket shopping consumers. They show that merchandising decisions should not be governed by accounting profits, but rather by a new construct they develop called marketing profits. Like us, they argue that simple techniques, based on readily available data, are

needed to guide decision making. However, there are some significant differences between their work and ours. In their model, each consumer type bases its store choice decision on the variety of a single category, what they call the lead category. Hence, expanding variety in category B has no influence on the store choice decision of category A lead customers. In contrast, our consumers base their decisions on the utility of multiple categories. As a result, there are minimal strategic interactions among categories in their model. A second key difference is how they improve decision making. They assume that a store makes optimal shelf space decisions and infer marketing profit parameters that would imply those decisions are optimal. They then use those marketing profit estimates to guide other merchandising decisions, such as advertising allocation. We use point-of-sales data to estimate basket profits and then derive optimal assortment decisions.

Assortment planning has attracted researchers from both operations and marketing fields. See Kök et al. (2006) for a recent review of this literature. van Ryzin and Mahajan (1999), Smith and Agrawal (2000), and Kök and Fisher (2004) study assortment selection and stocking decisions for a group of substitutable products in a single category assuming that store traffic is exogenous. Agrawal and Smith (2003) extend this work to the case with basket shopping consumers. Cachon et al. (2005) partially relaxes the exogenous store traffic assumption by considering consumer search behavior. The customers can choose to purchase an item at the store or to continue to search, which means that the fraction of "no-purchase" customers depends on the assortment. Chong et al. (2001) present an empirically-based modeling framework for managers to assess the revenue and lost sales implication of alternative assortments. Dreze et al. (1994) study the impact of shelf space on sales and Boatwright and Nunes (2001) study assortment reduction by making sure that certain attributes are represented in the assortment. Hopp and Xu (2005) study the impact of product modularity in the optimal product line length from a manufacturer's perspective. Hopp and Xu (2006) study price, service, and assortment competition in a single category between two retailers and find that the retailers provide less variety and lower prices in competition.

We use game theory to study competitive interactions in the decentralized regime and between the retailers. Gruca and Sudharshan (1991) and Basuroy and Nguyen (1998) study a market share game based on the multinomial logit (MNL) model and demonstrate that certain conditions are needed for equilibrium to exist. Karnani (1985) studies a multiplicative competitive interactions (MCI) model with several

firms that compete in a single market through several marketing decisions. Existence of equilibria is not guaranteed in his model because the profit function of a single firm is not jointly concave in the marketing variables. Our model has multiple customer types which may be considered as multiple markets. Monahan (1987) studies a model in which two firms compete with each other in several markets with an MCI model with a single marketing variable. Our model also has several markets, but the retailer's shares in different markets (customer types) are interdependent and multiple marketing variables (i.e., price and variety levels in all categories) play a role in each market.

2. Model Basics

Consider two retailers X and Y that carry two categories of goods. The set of products in category j is $\{1, 2, \dots, I\}$ for $j = 1, 2$, where I is a large number. Let subscript r denote retailer r , $r = x, y$. Retailer r offers n_{rj} products and sets its margin p_{rj} in category j . The unit procurement cost for product i in category j at retailer r is denoted β_{rji} . The selling price of a product to the consumers at retailer r is category margin plus the unit procurement cost of the product, i.e., $\beta_{rji} + p_{rj}$. The prices of each product are set such that the absolute margins for all products in a category are identical. This is actually the optimal pricing strategy of a firm: Anderson et al. (1992) prove that when customers follow an MNL choice model, optimal pricing policy for a group of products is an identical absolute mark-up policy. This pricing strategy is also observed in some retail settings. For example, different color/size shirts of the same style often have the same price tag.

The cost of providing an assortment with n_{rj} products is $C_{rj}(n_{rj})$, which is increasing, convex in n_{rj} , and can be parameterized with a scalar c_{rj} such that $\partial C_{rj}(c_{rj})/\partial c_{rj} > 0$. We describe a realistic replenishment system that yields convex costs in Appendix A.

The consumer choice model is based on a nested MNL framework. A consumer's utility from purchasing product i in category j at retailer r is $u_{rji} = v_{rji} - p_{rj} + \varepsilon$, where v_{rji} is the expected utility from the product less the unit cost of the product and ε is a random variable representing the heterogeneity of the utilities across consumers. We assume that ε are independently and identically distribute random variables following a Gumbel distribution with zero mean and variance $\pi^2 \mu^2 / 6$, i.e., $F(y) = \exp[-e^{-(y/\mu + \gamma)}]$, where γ is Euler's constant ($\gamma \approx 0.5722$). We assume that $\mu = 1$ for expositional simplicity. The products in each category at each retailer are indexed in descending order of their popularity, i.e., such that $v_{rj1} \geq v_{rj2} \geq \dots \geq v_{rjI}$.

There are three types of consumers in the market that are characterized by the contents of their shopping baskets: type 1 consumers would like to buy a

product in category 1 only, type 2 consumers would like to buy a product in category 2 only, type b consumers are basket shoppers and would like to buy a product from both categories. The number of consumers in the market for types 1, 2, and b are λ_1 , λ_2 , and λ_b , respectively. Consumers buy exactly one unit of one product in every category included in their basket. Consumers have perfect information about the offerings at both retail stores.

Consider the choice mechanism of a type j consumer for $j = 1, 2$. She has three options: retailer X , retailer Y , or no purchase. She chooses the alternative that gives her the maximum utility. The utility of the no-purchase option is u_{j0} , which follows a Gumbel distribution with mean v_{j0} and scale parameter 1. Let U_{rj} be a consumer's utility from the purchase of one variant at retailer r in category j , i.e., $U_{rj} = \max\{u_{rji}: 1 \leq i \leq n_{rj}\}$. Because the Gumbel distribution is closed under maximization, U_{rj} follows a Gumbel distribution with mean

$$E[U_{rj}] = \ln \sum_{1 \leq i \leq n_{rj}} e^{(v_{rji} - p_{rj})}$$

and scale parameter 1 (Ben-Akiva and Lerman 1985). The above computation is based on the assumption that retailer r offers the n_{rj} most popular products when choosing from $\{1, 2, \dots, I\}$. As shown in Proposition 9 in Appendix B, offering the most popular products is indeed the optimal strategy for a retailer, because in our model replacing a less popular product in the assortment with a more popular one increases the retailer's market share with no impact on costs. (van Ryzin and Mahajan 1999 prove the same result for a single category by explicitly modeling the inventory costs of each product.)

According to the nested-MNL model, the probability that a type j consumer chooses retailer r is

$$s_{rj} = \frac{\exp(E[U_{rj}])}{\exp(E[U_{xj}]) + \exp(E[U_{yj}]) + \exp(v_{j0})}. \quad (1)$$

Define the attractiveness function for each alternative as follows:

$$A_{rj} = e^{-p_{rj}} \sum_{i=1}^{n_{rj}} e^{v_{rji}} \quad \text{for } r = x, y,$$

$$Z_j = \exp(v_{j0}).$$

Note that A_{rj} is increasing and concave in n_{rj} and decreasing and convex in p_{rj} . Because of the no-purchase alternative in this share model, if both retailers increase prices or decrease variety, their total share decreases. We can rewrite each retailer's share among type j consumers:

$$s_{rj} = \frac{A_{rj}}{A_{xj} + A_{yj} + Z_j} \quad \text{for } r = x, y, \text{ and } j = 1, 2.$$

Now, consider a type b consumer. A basket shopping consumer chooses retailer r only if she prefers the assortment at r for both categories. As a result, the probability that a basket shopper chooses retailer r is

$$s_{rb} = s_{r1}s_{r2} \quad \text{for } r = x, y. \quad (2)$$

This is a multiplicative basket shopping model, as a retailer's share of basket shoppers is multiplicative in its share in each category. We discuss alternative demand models in  6, such as the additive utility model introduced by K ok (2003).

The profit of category j at retailer r is

$$\pi_{rj} = p_{rj}(\lambda_j s_{rj} + \lambda_b s_{rb}) - C_{rj}(n_{rj}).$$

For the rest of this paper, a variable name in bold letters denotes a vector of those variables: $\mathbf{n}_r = (n_{r1}, n_{r2})$, $\mathbf{p}_r = (p_{r1}, p_{r2})$, $\mathbf{A}_r = (A_{r1}, A_{r2})$, $\mathbf{c}_r = (c_{r1}, c_{r2})$, and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_b)$.

The variety decision \mathbf{n}_r is a vector of integers. Hereafter, we work with the continuous version of the problem because differentiability facilitates the derivation of the results and allows comparative statics. Also, we redefine

$$A_{rj} = e^{-p_{rj}} G_{rj}(n_{rj}),$$

where G_{rj} is a continuous, increasing, and concave function of n_{rj} , i.e., $G' > 0$ and $G'' \leq 0$. For fixed p_{rj} , A_{rj} is a continuous, increasing, and concave function of n_{rj} and there is a one-to-one correspondence between A_{rj} and n_{rj} . For convenience, we work with the attractiveness levels A_{rj} as the decision variables. Define $\tilde{C}_{rj}(A_{rj}) = C_{rj}(G_{rj}^{-1}(e^{p_{rj}} A_{rj}))$. We show that \tilde{C}_{rj} is an increasing convex function of attractiveness level in Proposition 10 in Appendix B. Rewriting the profit of category j at retailer r , we get

$$\pi_{rj} = p_{rj}(\lambda_j s_{rj} + \lambda_b s_{rb}) - \tilde{C}_{rj}(A_{rj}).$$

Although category profit is a function of \mathbf{A}_x , \mathbf{A}_y , \mathbf{n}_x , \mathbf{n}_y , \mathbf{p}_x , and \mathbf{p}_y , when referring to π_{rj} in the rest of this paper, we only include the variables that are being optimized at the moment. Finally, a *symmetric* game across categories (retailers) means that the data for all categories (retailers) are identical, i.e., λ_j , Z_j , and \tilde{C}_{rj} are the same for any $j(r)$. In a symmetric solution, the decisions are identical across categories (retailers).

3. Variety Competition with Fixed Prices

In this section, we take the retailers' prices as given and analyze their variety decisions. We start in  3.1 by characterizing the best response of one retailer given the variety choices of the other retailer. Section 3.2 presents our analysis of the competition between the retailers and  3.3 presents the basket profit heuristic as an alternative to CM.

3.1. Best Response of a Retailer

We study the best response of a retailer X with two management regimes: With CM, each category is managed as an independent unit. With OPT, a single decision maker manages both categories to maximize total profit.

3.1.1. Best Response of a Retailer with Category Management. The common practice of CM is an example of a decentralized regime for controlling assortment because each category manager is charged with maximizing profit with his or her assigned category. With CM, category manager j sets n_{xj} given the variety levels at Y and the variety level in the other category to maximize category j profit. The CM game between the two categories at retailer X is defined as

$$\max_{n_{xj}} \pi_{xj}(n_{xj} | n_{xk}, \mathbf{n}_y) \quad \text{s.t. } n_{xj} \geq 0, \\ \text{for } j = 1, 2, k \neq j. \quad (\text{CM})$$

Category managers do not actually need to know $\boldsymbol{\lambda}$ and the definition of share functions. We expect them to find the variety level that maximizes category profits (i.e., the best response function) given other categories' variety levels. They would estimate a demand function, say $d(n_{rj})$, as a proxy for $(\lambda_j s_{rj} + \lambda_b s_{rb})$ and then solve a single variable concave optimization problem to find n_{rj} that maximizes π_{rj} . Note that $d(n_{rj})$ depends on the variety levels of other categories.

CM can be interpreted as an explicit noncooperative game between the category managers because each category manager is responsible exclusively for the profits of her own category. Alternatively, it can be interpreted as an iterative application of single-category planning where each category's variety level is optimized assuming that all other assortment decisions for the retailer are fixed.

Before proceeding with our analysis of CM, we rewrite CM with decision variables \mathbf{A}_x :

$$\max_{A_{xj}} \pi_{xj}(A_{xj} | A_{xk}, \mathbf{A}_y) \quad \text{s.t. } A_{xj} \geq 0, \\ \text{for } j = 1, 2 \text{ and } k \neq j. \quad (\text{CM})$$

The best response of retailer X is determined by the equilibrium of the game between category managers at X . The next theorem characterizes the equilibria of CM. The following condition is needed for the second part of the theorem that establishes the uniqueness of the equilibrium:

$$4\lambda_1\lambda_2 \geq \lambda_b^2. \quad (\text{A1})$$

THEOREM 1. (i) CM is a supermodular game with a nonempty equilibrium set that has a largest and a smallest

element. The equilibria, denoted $\mathbf{A}_x^{\text{CM}}(\mathbf{A}_y)$ (equivalently $\mathbf{n}_x^{\text{CM}}(\mathbf{n}_y)$), are characterized by the solution to

$$p_{xj}\lambda_j \frac{\partial s_{xj}}{\partial A_{xj}} + p_{xj}\lambda_b \frac{\partial s_{xj}}{\partial A_{xj}} s_{xk} - \tilde{C}'_{xj}(A_{xj}) = 0$$

for $k \neq j$ and $j = 1, 2$. (3)

The largest (smallest) element is the Pareto best (worst) equilibrium. The largest and smallest equilibria are increasing in \mathbf{p}_x , $\boldsymbol{\lambda}$, and $-\mathbf{c}_x$. If $A_{xj} < A_{yj} + Z_j$ for all j , then the largest and smallest equilibria are decreasing in \mathbf{A}_y and \mathbf{n}_y .

(ii) If (A1) holds, then the equilibrium is unique.

(iii) If the categories are symmetric, then there exists only symmetric equilibria.

Supermodularity implies that the variety decisions across the two categories are strategic complements. This is not necessarily the case in every model with basket shoppers. (In the additive model presented in Kök (2003), categories can either be strategic complements or substitutes.) The variety level in each category is increasing in the retailer's margin and demand and decreasing in the cost coefficients. The variety level is decreasing in the competitor's variety level because it is more difficult to attract customers when the variety levels at the competitor are high. It is possible to have a unique equilibrium even if condition (A1) is not satisfied. Still, there are cases where CM has multiple equilibria. The third part of Theorem 1 rules out asymmetric equilibria in a symmetric game.

The supermodularity of CM has strong implications on the stability of equilibria. Specifically, tatonnement type algorithms (computing the best response of each category in an iterative manner) are guaranteed to converge to the largest or smallest equilibrium, depending on the starting point. Therefore, the centralized iterative optimization algorithm that maximizes one category's profits in each iteration is bound to converge to one of the CM equilibria. In cases with a unique equilibrium, the equilibrium point is a globally attracting equilibrium.

3.1.2. Best Response of a Centralized Retailer.

In the centralized system, total store profits at retailer X are maximized given the variety levels at retailer Y. We write this optimization problem with attractiveness levels of categories as decision variables:

$$\max_{\mathbf{A}_x} \sum_{j=1,2} \pi_{xj}(\mathbf{A}_x | \mathbf{A}_y) \quad \text{s.t. } A_{xj} \geq 0,$$

for $j = 1, 2$. (OPT)

Store profits are not jointly concave in general. We make the following assumption to ensure joint concavity:

$$4p_{x1}p_{x2}\lambda_1\lambda_2 \geq (p_{x1} + p_{x2})^2\lambda_b^2. \quad (\text{A2})$$

When the margins of the two categories are equal, $\lambda_j \geq \lambda_b$ for $j = 1, 2$ implies (A2).

THEOREM 2. (i) The optimal solution to OPT denoted $\mathbf{A}_x^{\text{OPT}}(\mathbf{A}_y)$ (equivalently $\mathbf{n}_x^{\text{OPT}}(\mathbf{n}_y)$) is increasing in \mathbf{p}_x , $\boldsymbol{\lambda}$, and $-\mathbf{c}_x$. If $A_{xj} < A_{yj} + Z_j$ for all j , then the optimal solution is decreasing in \mathbf{A}_y and \mathbf{n}_y .

(ii) If (A2) holds, then $\mathbf{A}_x^{\text{OPT}}(\mathbf{A}_y)$ is characterized by the unique solution to

$$p_{xj}\lambda_j \frac{\partial s_{xj}}{\partial A_{xj}} + (p_{xj} + p_{xk})\lambda_b \frac{\partial s_{xj}}{\partial A_{xj}} s_{xk} - \tilde{C}'_{xj}(A_{xj}) = 0$$

for $k \neq j$ and $j = 1, 2$. (4)

The marginal effect of A_{xj} on total profit is composed of its impact on own and cross-category sales. Hence, the first-order optimality conditions are based on the total profit earned from each customer type, whereas the CM solution is based on category profits.

3.1.3. Comparison of the Best Responses. If there were no basket shoppers, then CM and OPT would yield the same solution. Therefore, we assume that $\lambda_b > 0$. The following theorem establishes that CM never finds the optimal solution and provides less variety than the optimal solution.

THEOREM 3. A centralized retailer provides strictly more attractive categories (equivalently, higher variety in both categories) than a retailer with CM, i.e., $\mathbf{A}_x^{\text{OPT}} > \mathbf{A}_x^{\text{CM}}$ and $\mathbf{n}_x^{\text{OPT}} > \mathbf{n}_x^{\text{CM}}$.

The supermodularity and monotonicity results for the decentralized and the centralized solutions can be easily extended for more than two categories or for integer variety level variables.

We classify categories into two types: A *basket category* has a high co-occurrence rate in baskets with other categories. An *independent category* has most of its demand from single-category shoppers. We investigate whether basket categories or independent categories should be assigned more variety. Consider an extension of our model with three symmetric categories j, k , and l . Let λ_j, λ_k , and λ_l denote the demand rate of individual category shoppers and $\lambda_{\{j,k\}}$ and $\lambda_{\{j,l\}}$ denote the consumers with shopping baskets $\{j, k\}$ and $\{j, l\}$, respectively. Fix λ_j and suppose that $\lambda_k = \Lambda + \delta$, $\lambda_{\{j,k\}} = \Lambda - \delta$, $\lambda_l = \Lambda - \delta$, and $\lambda_{\{j,l\}} = \Lambda + \delta$, where $\delta \leq \Lambda$. If $\delta = 0$, categories k and l are identical; as δ increases, demand to each category remains constant, but l becomes more of a basket category.

By the implicit function theorem and the concavity of the profit function, $\partial A_k / \partial \delta$ has the same sign as the cross-partial

$$\frac{\partial^2}{\partial \delta \partial A_k} (\pi_{xj} + \pi_{xk} + \pi_{xl}) = ps'_k - 2ps'_k s_j.$$

The cross-partial is positive if and only if $s_j < 1/2$, implying that $\partial A_k^{\text{OPT}} / \partial \delta > 0$ and $\partial A_l^{\text{OPT}} / \partial \delta < 0$. Therefore, a centralized retailer with less than 50% market

share among nonbasket shoppers should assign more variety to individual categories and less variety to basket categories. The reverse is true for retailers with more than 50% market share. Similar analysis of CM yields that the cross-partial of the category profits with respect to A_k and δ are always positive, therefore, $\partial A_k^{\text{CM}}/\partial\delta > 0$ and $\partial A_l^{\text{CM}}/\partial\delta < 0$ holds for any market share. To summarize, both CM and OPT retailers prefer to assign more variety to individual categories when the retailer has less than half the market share among nonbasket shoppers. On the other hand, when the retailer has more than half the market share, the OPT retailer starts assigning more variety to basket categories while CM does the reverse.

3.2. Duopoly Results

In this section, we analyze the variety competition between two retailers. Two cases are of interest. In the first case, denoted CM-CM, both retailers employ category management. In the second case, denoted OPT-OPT, both retailers are under centralized management. We characterize the equilibrium in each case and compare the resulting variety levels and retailer profits.

3.2.1. Competition Between Retailers with Category Management. When both retailers are managed with CM, each category manager maximizes his or her category profits given the attractiveness levels of all other categories:

$$\max_{A_{rj}} \pi_{rj}(A_{rj} | A_{rk}, \mathbf{A}_q) \quad \text{s.t. } e^{v_{j0}} \geq A_{rj} \geq 0, \\ \text{for } j = 1, 2, k \neq j, r = x, y, q \neq r. \quad (\text{CM-CM})$$

The first-order conditions for equilibrium of CM-CM are the same as (3) only stated for both retailers.

The decision variables of the CM-CM game are $(\mathbf{A}_x, \mathbf{A}_y)$. We can redefine this game with variables $(\mathbf{A}_x, \mathbf{A}'_y)$, where $\mathbf{A}'_y = -\mathbf{A}_y$. Then, we can show that the category profit functions are supermodular in $(\mathbf{A}_x, \mathbf{A}'_y)$, when $A_{rj} < e^{v_{j0}}$ for all r and j . This condition implies that neither retailer can achieve more than 50% market share alone and that their combined share cannot exceed two thirds. Under this condition, we can characterize the equilibria of the game as follows.

If (A1) holds, by part (ii) of Theorem 1, \mathbf{A}_r^{CM} is a continuous decreasing function of \mathbf{A}_q for $q \neq r$. Therefore, there exists a unique equilibrium of the CM-CM game with symmetric categories and retailers. In a symmetric equilibrium, the equilibrium conditions are identical for retail-category combinations, therefore, we drop retailer subscripts r and j . That is, $p_{rj} = p$, $\lambda_j = \lambda$, $Z_j = Z$, $\tilde{C}_{rj} = \tilde{C}$, and $A_{rj} = A$. Define $\delta = A + Z$ and $\phi = A + A + Z$.

THEOREM 4. (i) CM-CM is a supermodular game in $(\mathbf{A}_x, -\mathbf{A}_y)$. There exists a largest and a smallest equilibrium that increase with $(\mathbf{p}_x, -\mathbf{p}_y)$ and $(-\mathbf{c}_x, \mathbf{c}_y)$.

(ii) If (A1) holds and categories and retailers are symmetric, then there exists a unique symmetric equilibrium of the CM-CM game $\mathbf{A}^{\text{CM-CM}} = (\mathbf{A}^{\text{CM}}, \mathbf{A}^{\text{CM}})$ that is characterized by

$$p\lambda\delta\phi^{-2} + p\lambda_y\delta\phi^{-3}A - \tilde{C}'(A) = 0. \quad (5)$$

3.2.2. Competition Between Centralized Retailers. When both retailers are under OPT, each retailer maximizes its total store profits given the attractiveness levels of the other retailer:

$$\max_{A_r} \sum_{j=1,2} \pi_{rj}(\mathbf{A}_r | \mathbf{A}_q) \quad \text{s.t. } e^{v_{j0}} \geq A_{rj} \geq 0, \\ \text{for } j = 1, 2, r = x, y, q \neq r. \quad (\text{OPT-OPT})$$

The first-order conditions for equilibrium of OPT-OPT are the same as (4) only stated for both retailers.

Similar to the CM-CM game, we can redefine OPT-OPT with decision variables $(\mathbf{A}_x, -\mathbf{A}_y)$ and show that the game is supermodular. If (A2) holds, by Theorem 2, $\mathbf{A}_r^{\text{OPT}}$ is a continuous decreasing function of \mathbf{A}_q for $q \neq r$. Therefore, there exists a unique equilibrium of the OPT-OPT game with symmetric categories and retailers. The following theorem characterizes the equilibria of the game.

THEOREM 5. (i) OPT-OPT is a supermodular game in $(\mathbf{A}_x, -\mathbf{A}_y)$. There exists a largest and a smallest equilibrium that increase with $(\mathbf{p}_x, -\mathbf{p}_y)$ and $(-\mathbf{c}_x, \mathbf{c}_y)$.

(ii) If (A2) holds and categories and retailers are symmetric, then there exists a unique symmetric equilibrium of the OPT-OPT game $\mathbf{A}^{\text{OPT-OPT}} = (\mathbf{A}^{\text{OPT}}, \mathbf{A}^{\text{OPT}})$ that is characterized by

$$p\lambda\delta\phi^{-2} + (2p)\lambda_y\delta\phi^{-3}A - \tilde{C}'(A) = 0.$$

Part (ii) of Theorems 4 and 5 can be slightly generalized by not requiring categories to be symmetric as long as retailers are symmetric.

3.2.3. Comparison of Equilibria. The following theorem compares the unique symmetric equilibrium of CM-CM with that of OPT-OPT when categories and retailers are symmetric.

THEOREM 6. If (A2) holds, then

- (i) $\mathbf{A}^{\text{CM-CM}} < \mathbf{A}^{\text{OPT-OPT}}$ and $\mathbf{n}^{\text{CM-CM}} < \mathbf{n}^{\text{OPT-OPT}}$,
- (ii) $\pi(\mathbf{A}^{\text{CM-CM}}) \geq \pi(\mathbf{A}^{\text{OPT-OPT}})$.

This theorem shows that CM retailers provide less variety than the centralized retailers in equilibrium. The first part of the theorem can be generalized to asymmetric categories and retailers, as the proof does not require symmetry assumptions.

In OPT-OPT, because of the competition, retailers offer more attractive categories than they would if they colluded. In the CM-CM game, however, the lack of coordination between categories in CM reduces variety and balances some of the effect of competition.

Consider the management regime choice game (CM or OPT) between the retailers. In this two-by-two normal form game, a prisoner’s dilemma type of situation arises: The retailer profits are higher in the CM-CM game than the OPT-OPT game. Nevertheless, each retailer has an incentive to switch to centralized management because OPT maximizes the store profits given the other retailer’s variety levels. OPT is the dominant strategy for each retailer. As a result, the Nash equilibrium of the regime choice game is OPT-OPT.

This dilemma arises in all competitive games. Take differentiated Bertrand price competition, for example. It is clear that in equilibrium players charge lower prices and generate lower profits compared to joint profit maximization. Therefore, if both players can adopt a pricing policy that is less aggressive (analogous to CM in our case) than their best response (OPT in our case), they can generate higher profits. However, given that we are analyzing a competitive scenario, collusion is not possible and the player’s correct strategy is to follow its best response.

3.3. Category Management with Basket Profits

In this subsection, we discuss the extension of our results to settings with more than two categories and propose a heuristic solution that provides near-optimal solutions with decentralized regimes. Suppose that retailers carry more than two categories indexed by $j \in J = \{1, \dots, N\}$. The notation for cost, price, variety, and attractiveness variables carries over. However, now there can be more than three types of consumers in the market. Specifically, consumers are characterized by the contents of their shopping baskets or the shopping list. Let t be the index of the elements of the power set of J , $\{B: B \subseteq J\}$. $B_t \subset J$ denotes a shopping basket that contains categories $j \in B_t$. The number of type t consumers in the market is λ_t . A consumer of type t purchases exactly one unit of a product from each category in her shopping list B_t . Similar to the share function (2) for basket shoppers, the probability of a type t consumer choosing retailer r is given by

$$s_r^t = \prod_{j \in B(t)} s_{rj} \quad \text{for } r = x, y.$$

The expected profit in category j is

$$\pi_{rj} = p_{rj} \sum_{t|j \in B_t} \lambda_t s_r^t - \tilde{C}_{rj}(A_{rj}).$$

A retailer’s best response with centralized and decentralized regimes are defined by (CM) and (OPT) with the modification that j is now defined over J . All our results for CM and OPT (Theorems 1 through 3) except for the uniqueness results extend to the $N > 2$ case.

The CM solution $\mathbf{A}_x^{\text{CM}}(\mathbf{A}_y)$ emerges as the outcome of a natural iterative process: category managers set variety levels, store traffic and sales are realized, category managers reassess the demand function for their category and choose new assortments, etc. The same process occurs if the retailer does category by category optimization in an iterative manner. The process always converges to one of the CM equilibria because of the supermodularity of the game. Despite its simplicity, we show in the numerical results section that \mathbf{A}_x^{CM} can significantly deviate from the optimal solution $\mathbf{A}_x^{\text{OPT}}$. On the other hand, it is not easy to implement the centralized optimal solution: it involves estimating the number of customers for all 2^N basket types and an N -dimensional optimization of a function that is not necessarily well behaved.

We now introduce a decentralized heuristic that promises the best of both worlds (i.e., the simplicity of CM and a profit level close to that of the optimal solution). CM’s main weakness is that it fails to recognize the intercategory effects of variety decisions while underestimating the marginal benefit of variety. From the perspective of the manager of category j , an additional sale is only worth p_j , but from the retailer’s perspective it is worth more. We approximate the true marginal benefit to the retailer with the weighted average of the gross margins across basket types. We call this new metric *basket profit* because it measures the total profit earned from a customer. Specifically, let \hat{p}_{xj} be the basket profit from category j :

$$\hat{p}_{xj} = \sum_{t|j \in B_t} \left(\lambda_t \sum_{l \in B_t} p_{xl} \right) / \sum_{t|j \in B_t} \lambda_t. \quad (6)$$

The manager for category j at retailer X then maximizes the following profit function:

$$\hat{\pi}_{xj} = \hat{p}_{xj} \sum_{t|j \in B_t} \lambda_t s_x^t - \tilde{C}_{xj}(A_{xj}).$$

Each CM then chooses the attractiveness (or variety) level of her own category,

$$\begin{aligned} \max_{A_{xj}} \hat{\pi}_{xj}(A_{xj} | A_{x1}, \dots, A_{x,j-1}, A_{x,j+1}, \dots, A_{xN}, \mathbf{A}_y) \\ \text{s.t. } A_{xj} \geq 0, \quad \text{for } j \in J \text{ and } k \neq j. \end{aligned} \quad (\text{CM-B})$$

Note that all our results on the CM equilibria apply directly to the CM-B heuristic. Comparing the profit functions with CM-B and OPT, we see that CM-B uses a weighted average \hat{p}_{xj} for all consumer types instead

Table 1 Attachment Rate of the Television Category at an Electronics Retailer with Other Categories

	Basket of television category										Total
Category	282	602	383	281	321	384	190	389	604	376	
Attachment rate (%)	3.9	3.7	3.7	3.5	2.8	2.8	2.6	2.3	2.2	2.1	29.6

of using $\sum_{k \in B_t} p_{xk}$ for consumer type t . We test the effectiveness of this heuristic in  5 and show that it yields near-optimal results.

From a practical perspective, note that the basket profit for each category is easily computed from point of sale (POS) data: it is the average gross margin earned from customers who purchased a basket including that category. Alternatively, basket profit from category j can be computed by using a construct called attachment rate. The attachment rate of categories j and k , a_{jk} , is the percentage of category j customers who also bought from category k :

$$a_{jk} = \frac{\sum_{t | \{j, k\} \in B_t} \lambda_t}{\sum_{t | j \in B_t} \lambda_t}.$$

It is possible to compute \hat{p}_{xj} with this data as follows:

$$\hat{p}_{xj} = p_{xj} + \sum_{k \neq j} p_{xk} a_{jk}. \quad (7)$$

It can be verified that (6) and (7) are equivalent. However, the use of (7) can be more convenient because attachment rate is a metric that is currently used by some retailers. Consider the example for the television category at an electronics retail chain (Circuit City) in Table 1. The table shows the attachment rate of this category with other categories.

4. Price and Variety Competition

This section analyzes the competition between retailers with endogenous prices. Retailers choose prices and variety levels in each category to optimize store or category profits depending on the management regime. Again, we analyze the CM-CM and OPT-OPT games. In both cases, we show that the interaction between categories (within or across retailers) depends on the price and the variety level of each category only through an aggregate attractiveness function A . For any given vector of attractiveness levels, each category can choose its optimal price and variety level independent of other categories and independent of the management regime. Therefore, we consider the attractiveness levels in the analysis of the games. Then, for any given equilibrium, we compute the variety level and price of each category.

The CM-CM game is defined as follows. For $j = 1, 2, k \neq j, r = x, y, q \neq r$,

$$\begin{aligned} & \max_{p_{rj}, n_{rj}} \pi_{rj}(p_{rj}, n_{rj} | p_{rk}, n_{rk}, \mathbf{p}_q, \mathbf{n}_q) \\ & \text{s.t. } A_{rj} = G(n_{rj})e^{-p_{rj}}. \end{aligned}$$

Equivalently,

$$\begin{aligned} & \max_{p_{rj}, A_{rj}} \pi_{rj}(p_{rj}, A_{rj} | p_{rk}, A_{rk}, \mathbf{p}_q, \mathbf{A}_q) \\ & \text{s.t. } n_{rj} = G^{-1}(A_{rj}e^{p_{rj}}). \end{aligned}$$

We omit the expression for n_{rj} hereafter. Because π_{rj} does not depend on the prices of the other categories at any retailer, we have

$$\max_{p_{rj}, A_{rj}} \pi_{rj}(p_{rj}, A_{rj} | A_{rk}, \mathbf{A}_q).$$

Rewriting the problem for each category as a two-stage optimization problem,

$$\begin{aligned} & \max_{A_{rj}} \max_{p_{rj}} \pi_{rj}(p_{rj}, A_{rj} | A_{rk}, \mathbf{A}_q) \\ & = \max_{A_{rj}} \pi_{rj}(p_{rj}^*(\mathbf{A}_r, \mathbf{A}_q) | A_{rk}, \mathbf{A}_q), \quad (\text{P:CM-CM}) \end{aligned}$$

where

$$p_{rj}^*(\mathbf{A}_r, \mathbf{A}_q) = \arg \max_{p_{rj}} \pi_{rj}(p_{rj} | \mathbf{A}_r, \mathbf{A}_q), \quad (8)$$

$$n_{rj}^*(\mathbf{A}_r, \mathbf{A}_q) = G^{-1}(A_{rj}e^{p_{rj}^*}). \quad (9)$$

The above derivation leads to two important observations. First, the interaction between categories and retailers is through the attractiveness levels A_{rj} only (a composite variable of price and variety levels). Second, given the attractiveness levels of all categories, the price optimization is separable across categories. That is, each category profit is independent of the prices in other categories. Given $(\mathbf{A}_r, \mathbf{A}_q)$, both the price and variety level of each category rj are uniquely determined from Equations (8) and (9).

Next, we show that OPT-OPT game with endogenous pricing possesses the same properties:

$$\begin{aligned} & \max_{\mathbf{p}_r, \mathbf{A}_r} \sum_j \pi_{rj}(\mathbf{p}_r, \mathbf{A}_r | \mathbf{p}_q, \mathbf{A}_q) \\ & = \max_{\mathbf{p}_r, \mathbf{A}_r} \sum_j \pi_{rj}(p_{rj}, \mathbf{A}_r | \mathbf{A}_q) \\ & = \max_{\mathbf{A}_r} \sum_j \max_{p_{rj}} \pi_{rj}(p_{rj}, \mathbf{A}_r | \mathbf{A}_q) \\ & = \max_{\mathbf{A}_r} \sum_j \pi_{rj}(p_{rj}^*(\mathbf{A}_r, \mathbf{A}_q), \mathbf{A}_r | \mathbf{A}_q). \quad (\text{P:OPT-OPT}) \end{aligned}$$

Again, the interaction between categories are through attractiveness levels only. Moreover, given $(\mathbf{A}_r, \mathbf{A}_q)$, the price and the variety levels in each category are uniquely determined by the same set of

equations in CM-CM, namely, (8) and (9). Thus, given attractiveness levels, the optimal price and the variety levels in P:OPT-OPT and P:CM-CM are identical. This property simplifies the analysis of the duopoly game significantly.

For analytical tractability, we assume that the cost of variety is linear and candidate products in a category are equally popular.¹ That is,

$$C_{rj}(n_{rj}) = c_{rj}n_{rj} \quad \text{and} \quad G(n_{rj}) = gn_{rj} \quad \text{for all } r, j. \quad (\text{A3})$$

In this case, the optimal price and variety levels in each category given $(\mathbf{A}_r, \mathbf{A}_q)$ are given by the following. Define $D_{rj} = \lambda_j s_{rj} + \lambda_b s_{rb}$:

$$\begin{aligned} p_{rj}^*(\mathbf{A}_r, \mathbf{A}_q) &= \arg \max_{p_{rj}} p_{rj} D_{rj} - C_{rj}(G^{-1}(e^{p_{rj}} A_{rj})) \\ &= \arg \max_{p_{rj}} p_{rj} D_{rj} - (c_{rj}/g)e^{p_{rj}} A_{rj} \\ &= \ln(g/c_{rj}) + \ln D_{rj} - \ln A_{rj}, \\ n_{rj}^*(\mathbf{A}_r, \mathbf{A}_q) &= D_{rj}/c_{rj}. \end{aligned}$$

Substituting these in the category profit function, we obtain

$$\pi_{rj}(p_{rj}^*(\mathbf{A}_r, \mathbf{A}_q), \mathbf{A}_r | \mathbf{A}_q) = (p_{rj}^* - 1)D_{rj}.$$

Because we are interested in cases where profits are nonnegative, hereafter we assume that $p_{rj}^* \geq 1$. We now characterize the best response of a retailer in (P:CM-CM) and in (P:OPT-OPT). \mathbf{A}_r^{CM} and $\mathbf{A}_r^{\text{OPT}}$ denote the solution to (P:CM-CM) and (P:OPT-OPT), respectively.

THEOREM 7. *Best response of a retailer r in (P:CM-CM) and (P:OPT-OPT) facing \mathbf{A}_q where $(A_{q1} = A_{q2})$: If (A3) holds and $A_{rj} \leq Z_j$, then*

- (i) $\mathbf{A}_r^{\text{CM}}(\mathbf{A}_q)$ and $\mathbf{A}_r^{\text{OPT}}(\mathbf{A}_q)$ are decreasing in \mathbf{A}_q , λ , and \mathbf{c}_r ,
- (ii) $\mathbf{A}_r^{\text{CM}}(\mathbf{A}_q)$ is the unique equilibrium response if $\lambda_j \lambda_k > \lambda_b^2$,
- (iii) $\mathbf{A}_r^{\text{CM}}(\mathbf{A}_q) < \mathbf{A}_r^{\text{OPT}}(\mathbf{A}_q)$,
- (iv) $\mathbf{n}_r^*(\mathbf{A}_r^{\text{CM}}(\mathbf{A}_q), \mathbf{A}_q) < \mathbf{n}_r^*(\mathbf{A}_r^{\text{OPT}}(\mathbf{A}_q), \mathbf{A}_q)$,
- (v) If $\lambda_j \geq \lambda_b$, then $\mathbf{p}_r^*(\mathbf{A}_r^{\text{CM}}(\mathbf{A}_q), \mathbf{A}_q) > \mathbf{p}_r^*(\mathbf{A}_r^{\text{OPT}}(\mathbf{A}_q), \mathbf{A}_q)$ in symmetric solutions (i.e., $A_{rj} = A_{rk}$).

This theorem shows that most of our results on the best response of a retailer in variety competition extends to the endogenous prices case. Supermodularity of the payoff functions and the monotonicity in demand and cost parameters are established in part (i). Further, in the best response of a retailer, the attractiveness levels of its categories are decreasing in the attractiveness of the competitor's categories. For

¹ For general concave G and convex C functions, we were not able to show unimodality of π_{rj} , which is a sufficient condition to guarantee the existence of equilibria. However, we found π_{rj} to be unimodal in all numerical examples we tested.

CM, we can also show that there is a unique equilibrium to the CM game with pricing that determines the retailer's best response. However, we cannot show joint concavity of OPT with pricing.

Parts (iii) and (iv) indicate that with CM a retailer provides less attractive categories and less variety than a centralized retailer. The impact of an increase in A_{rj} on p_{rj} is negative, while the impact of an increase in A_{rk} is positive, therefore a definite comparison of prices in CM and OPT cannot be made in general. However, part (v) shows that in symmetric solutions, a retailer with CM charges lower prices than a centralized retailer. To summarize, a retailer with CM offers lower variety and higher prices, leading to less attractive categories overall.

The next theorem summarizes our results on duopoly price and variety competition. Let $\mathbf{A}^{\text{CM-CM}}$ denote the equilibrium solution to (P:CM-CM), and $\mathbf{n}^{\text{CM-CM}}$ and $\mathbf{p}^{\text{CM-CM}}$ denote the optimal variety and price levels at the equilibrium. Similarly, let $\mathbf{A}^{\text{OPT-OPT}}$, $\mathbf{n}^{\text{OPT-OPT}}$, and $\mathbf{p}^{\text{OPT-OPT}}$ denote the solution to (P:OPT-OPT).

THEOREM 8. *Consider the duopoly games (P:CM-CM) and (P:OPT-OPT) restricted to symmetric strategies, i.e., $A_{r1} = A_{r2}$ for all r . If (A3) holds and $A_{rj} \leq Z_j$, then*

- (i) (P:CM-CM) is a supermodular game in $(\mathbf{A}_x, -\mathbf{A}_y)$ and (P:OPT-OPT) is a supermodular game in $(\mathbf{A}_x, -\mathbf{A}_y)$.
- (ii) There exist a largest and a smallest equilibria in both duopoly games. In CM-CM, there exists a unique symmetric equilibrium if all categories are symmetric.

(iii) Assuming a symmetric solution characterized by the first-order conditions in each case: $\mathbf{A}^{\text{CM-CM}} < \mathbf{A}^{\text{OPT-OPT}}$, $\mathbf{n}^{\text{CM-CM}} < \mathbf{n}^{\text{OPT-OPT}}$. If $\lambda_j \geq \lambda_b$, then, $\mathbf{p}^{\text{CM-CM}} > \mathbf{p}^{\text{OPT-OPT}}$, $\pi(\mathbf{A}^{\text{CM-CM}}) \geq \pi(\mathbf{A}^{\text{OPT-OPT}})$.

The third part of the theorem states that the variety level and the overall attractiveness levels are lower, similar to the variety competition case, and the prices and category profits are higher in the CM-CM equilibrium than the OPT-OPT equilibrium.

4.1. Category Management with Basket Profits

The basket profits heuristic introduced in §3.3 enabled us to achieve near-optimal solutions in a decentralized regime by using basket profits instead of category profits. Because the prices (or margins) are endogenous in the price and variety competition case, the heuristic cannot be directly applied and requires some modification.

Given the price vector at retailer X , basket profit can be computed using Equation (7). However, p_{rj} is now a decision variable and therefore \hat{p}_{xj} is no longer a constant. In CM with basket profits, each category manager faces the following problem:

$$\max_{p_{xj}, n_{xj}} \hat{\pi}_{xj} = \left(p_{xj} + \sum_{k \neq j} p_{xk} a_{jk} \right) \sum_{t | j \in B_t} \lambda_t s_{xt}(\mathbf{A}_x, \mathbf{A}_y) - \tilde{C}_{xj}(A_{xj})$$

for all j . (P:CM-B)

Some clarifications are in order. First, (P:CM-B) uses basket profits while computing profits given shares, but computes the attractiveness of categories (and market shares among different customer types) using category profits. That is, both CM and (P:CM-B) use p_{xj} in computing A_{rj} and s_r^t . Second, (P:CM-B) is a different game from CM because decision variables for other categories (i.e., p_{xk}) are now included in the basket profit of category j . Therefore, the game is more involved and analytical properties of CM do not necessarily carry over. We found in our numerical experiments that the iterative process converged to an equilibrium in all cases. We report the performance of this heuristic in the next section.

5. Numerical Results

The numerical study in this section is composed of two parts. The first part focuses on variety competition with fixed prices and the performance of the basket profits heuristic. The second part focuses on price and variety competition and the performance of the basket heuristic in comparison to CM.

5.1. Variety Competition

Table 2 presents the best response of a retailer with CM and OPT. For cases with multiple equilibria in CM, we only report (the largest and the best) equilibrium. All categories are symmetric, costs are linear, and attractiveness functions are linear in the variety level (i.e., $A_{xj} = n_{xj}$). As seen from the table, CM provides, on average, 26% less variety than OPT. The profit loss due to CM ranges from 0% to 100% with an average of 13%.

Table 2 Variety Competition. Best Response of a Retailer with CM and OPT. Two Symmetric Categories, $\rho_j = 1$

λ_j	λ_b	A_{yj}	Z_j	c_j	A_{xj}^{CM}	A_{xj}^{OPT}	π^{CM}	π^{OPT}	$1 - \pi^{CM}/\pi^{OPT}$ (%)
20	80	5	15	0.5	32.8	54.0	53.8	60.4	11
20	80	15	15	0.5	30.0	56.4	30.0	37.9	21
20	80	5	15	1	11.2	27.4	12.6	21.8	42
20	80	15	15	1	0.0	20.5	0.0	1.6	100
20	80	5	15	2	0.0	0.0	0.0	0.0	0
20	80	15	15	2	0.0	0.0	0.0	0.0	0
50	50	5	15	0.5	37.5	49.7	70.2	72.5	3
50	50	15	15	0.5	38.4	52.6	49.3	51.6	5
50	50	5	15	1	18.5	26.2	34.1	36.5	6
50	50	15	15	1	14.6	22.9	14.3	16.2	12
50	50	5	15	2	4.2	8.1	3.6	4.7	24
50	50	15	15	2	0.0	0.0	0.0	0.0	0
80	20	5	15	0.5	41.1	45.7	84.6	84.9	0
80	20	15	15	0.5	44.3	49.3	65.3	65.6	0
80	20	5	15	1	22.6	25.2	50.9	51.2	1
80	20	15	15	1	21.5	24.2	30.8	31.0	1
80	20	5	15	2	9.4	10.6	17.6	17.8	1
80	20	15	15	2	5.3	6.1	3.7	3.8	1

Table 3 Effect of Problem Parameters in Variety Competition. Best Response of a Retailer with CM and OPT. Two Symmetric Categories, $\rho_j = 1$

	$1 - \pi^{CM}/\pi^{OPT}$ (%)	$1 - n^{CM}/n^{OPT}$ (%)
(λ_j, λ_b)		
(20, 80)	1	10
(50, 50)	6	29
(80, 20)	42	60
Competition ($A_{yj} + Z_j$)		
20	13	31
30	22	37
Total cost ($\sum_j c_j$)		
1	7	26
1.5	14	31
4.5	30	43

Table 3 illustrates the effect of problem parameters on the deviation in variety level and profit loss. The proportion of basket shoppers in this data set is 80%, 50%, and 20% when (λ_j, λ_b) is (20, 80), (50, 50), and (80, 20), respectively. As one can expect, the loss due to CM is larger with a higher proportion of basket shoppers, higher cost of providing variety (c_j), and stronger competition (higher $A_{yj} + Z_j$).

Table 4 compares the results of CM and the basket profits heuristic (CM-B) with the best response of a retailer with OPT. The data set consists of 12 symmetric and 24 asymmetric category instances for $N = 2$, and 18 symmetric category examples for $N = 3$. CM-B, denoted with superscript B in the table, reduces the average profit loss of CM from 21% to 2%. CM-B works equally well for asymmetric and symmetric scenarios. In the $N = 3$ case, the performance of CM-B with respect to OPT is worse than the $N = 2$ case, however, the performance of CM is even worse. In all cases, CM-B recovers more than 85% of the profit loss due to CM.

One drawback of the basket profits heuristic is that like CM, it can converge to a bad equilibrium in the case of multiple equilibria, especially when all categories have zero variety in the worst equilibrium. In the 26 instances for $N = 2$, there were three cases where CM had multiple equilibria. CM-B had a unique equilibrium in all 26 instances, which eliminated the possibility of a worst case. In general, if

Table 4 Basket Profits Heuristic in the Best Response of a Retailer in Variety Competition

N	Data	Average profit loss (%)		Average deviation in variety levels (%)	
		$1 - \pi^{CM}/\pi^{OPT}$	$1 - \pi^B/\pi^{OPT}$	$1 - n^{CM}/n^{OPT}$	$1 - n^B/n^{OPT}$
2	Symmetric	16.9	2.6	33.7	-8.7
	Asymmetric	15.9	0.4	32.5	-7.1
3	Symmetric	32.4	4.8	47.8	-12.1

Table 5 Variety Competition. Comparison of Equilibria with CM or OPT. Symmetric Categories, $\rho_j = 1$

λ_j	λ_b	Z_j	c_j	A^{CM-CM}	$A^{OPT-OPT}$	π^{CM-CM}	$\pi^{OPT-OPT}$
50	50	30	0.5	29.5	40.0	14.6	9.6
50	50	30	1	9.1	13.1	4.2	2.5
50	50	30	2	0.0	0.0	0.0	0.0
50	50	60	0.5	18.2	26.2	4.2	2.5
50	50	60	1	0.0	0.0	0.0	0.0
50	50	60	2	0.0	0.0	0.0	0.0
80	20	30	0.5	40.5	44.5	23.2	21.0
80	20	30	1	16.5	18.0	11.7	10.6
80	20	30	2	3.5	3.9	1.5	1.3
80	20	60	0.5	32.9	36.1	11.7	10.6
80	20	60	1	7.1	7.7	1.5	1.3
80	20	60	2	0.0	0.0	0.0	0.0

there are multiple equilibria, starting with a high variety level when a category is first introduced ensures that CM-B (or CM) converges to the best equilibrium.

Table 5 presents the equilibrium results in the duopolistic variety competition. The variety levels are, on average, 11% lower in the CM-CM competition than OPT-OPT.

5.2. Price and Variety Competition

Table 6 presents the best response of a retailer with CM and OPT. Similar to the variety competition, CM provides less attractive categories and results in an average profit loss of 21%. We can see that variety levels are lower and prices are higher with CM in all examples. The variety level with CM is 31% less, and prices with CM are 6% higher compared to those with OPT.

Table 7 compares the performance of the basket profits heuristic (P:CM-B) with the best response of a retailer with OPT. The data set is the same as Table 6. P:CM-B, denoted with superscript B in the table, achieves near-optimal solutions in almost all instances except for one case. The average profit loss was 4%, whereas the average loss due to CM was 21%. It can be seen from Tables 4 and 7 that the

gap between the heuristic and the optimal solution increases as the number of categories and the cost of providing variety increase or the market share of the retailer decreases. It can also be observed that the gap is higher with a medium proportion of basket shoppers. This is due to the fact that CM-B is optimal when there are no basket shoppers or no individual category shoppers.

Table 8 presents the equilibrium results in the duopolistic price and variety competition. The attractiveness levels are, on average, 19% lower in CM-CM competition than OPT-OPT. The variety levels are, on average, 7% lower in CM-CM than OPT-OPT. The prices are, on average, 12% higher in CM-CM than OPT-OPT.

To summarize, the numerical study confirmed our analytical findings on the comparison of different management regimes, showed that the profit impact of the management regime can be significant, and demonstrated that the basket-profit heuristic can be an effective tool to coordinate categories in all settings.

6. Additive Basket Utility and Cherry-Picking Consumers

As mentioned earlier in §2, the multiplicative choice behavior defined by (2) can be replaced with alternative models to refine the consumer choice model. A basket shopper may choose a retailer when her total expected utility at that retailer is higher than other alternatives, even if the retailer is not her first choice in both categories. Kök (2003) develop an additive utility demand model for basket shoppers. In the additive model, a basket shopping consumer's total utility at retailer r is $U_{r1} + U_{r2}$ and the consumer chooses the retailer with the maximum total expected utility. Kök (2003) develop an approximate closed-form formula for the share equations and analyze the best response of a retailer with centralized and decentralized management. They establish that

Table 6 Price and Variety Competition. Best Response of a Retailer with CM or OPT. Two Symmetric Categories

λ_j	λ_b	A_{yj}	Z_j	c_j	A_{xj}^{CM}	A_{xj}^{OPT}	p^{*CM}	p^{*OPT}	π^{*CM}	π^{*OPT}	π^{CM}	π^{OPT}	$1 - \pi^{CM}/\pi^{OPT}$ (%)
40	160	5	15.0	0.5	16.4	36.0	1.82	1.63	100.8	183.8	82.5	115.6	29
40	160	15	15.0	0.5	12.3	37.8	1.41	1.34	50.1	144.2	20.5	48.7	58
40	160	5	15	1.0	0.0	16.9	0.69	1.12	0.0	52.0	0.0	12.6	100
40	160	15	15	1.0	0.0	0.0	0.29	0.29	0.0	0.0	0.0	0.0	0
100	100	5	15	0.5	20.2	30.8	2.01	1.84	151.1	194.8	152.8	164.5	7
100	100	15	15	0.5	21.3	33.4	1.71	1.57	117.3	160.9	83.1	92.0	10
100	100	5	15.0	1.0	9.9	16.3	1.49	1.38	44.0	65.0	43.4	50.0	13
100	100	15	15.0	1.0	5.4	11.0	1.18	1.13	17.6	34.1	6.4	8.8	28
160	40	5	15	0.5	22.8	26.4	2.14	2.06	193.0	208.0	219.6	221.3	1
160	40	15	15	0.5	25.7	29.8	1.86	1.79	164.7	179.3	141.2	142.5	1
160	40	5	15	1.0	13.3	15.4	1.66	1.61	70.3	77.2	93.4	94.4	1
160	40	15	15	1.0	12.1	14.0	1.40	1.37	49.4	55.1	39.9	40.4	1

Table 7 Basket Profits Heuristic in the Best Response of a Retailer in Price and Variety Competition. Two Symmetric Categories

λ_j	λ_b	A_{yj}	Z_j	c_j	A_{xj}^{CM}	A_{xj}^B	A_{xj}^{OPT}	π^{CM}	π^B	π^{OPT}	$1 - \pi^{CM}/\pi^{OPT}$ (%)	$1 - \pi^B/\pi^{OPT}$ (%)
40	160	5	15.0	0.5	16.4	37.9	36.0	82.5	115.4	115.6	29	0
40	160	15	15.0	0.5	12.3	41.1	37.8	20.5	48.3	48.7	58	1
40	160	5	15	1.0	0.0	19.7	16.9	0.0	12.0	12.6	100	5
40	160	15	15	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0	0
100	100	5	15	0.5	20.2	34.0	30.8	152.8	163.7	164.5	7	0
100	100	15	15	0.5	21.3	38.4	33.4	83.1	90.9	92.0	10	1
100	100	5	15.0	1.0	9.9	19.8	16.3	43.4	48.5	50.0	13	3
100	100	15	15.0	1.0	5.4	17.6	11.0	6.4	5.8	8.8	28	34
160	40	5	15	0.5	22.8	28.5	26.4	219.6	220.8	221.3	1	0
160	40	15	15	0.5	25.7	33.0	29.8	141.2	141.9	142.5	1	0
160	40	5	15	1.0	13.3	17.5	15.4	93.4	93.6	94.4	1	1
160	40	15	15	1.0	12.1	17.1	14.0	39.9	39.2	40.4	1	3

a retailer with CM provides less variety than the optimal. Most of our results (specifically, the results in variety competition and the results on the best response of a retailer in price and variety competition) can be shown to hold with the additive utility model but require more restrictive conditions. For example, the analog of Theorem 1 (i.e., supermodularity of CM) holds only when $A_{rj} \leq e^{\eta_0}$.

Consumers may also split their shopping between two alternatives (e.g., buy 1 at X and buy 2 at Y or buy 1 at X and don't buy 2) if the benefit of doing so exceeds the disutility of having to visit more than one store. This is called cherry-picking. A basket shopping consumer evaluates the total utility associated with different alternatives and chooses the one that maximizes her total basket utility:

$$\max_{r, q \in \{x, y, z\}} U_{r1} + U_{q2} - \eta 1\{r = q\},$$

where η is the disutility of visiting an extra store, and $1\{r = q\}$ is an indicator function equal to one if $r = q$

and zero otherwise. Clearly, the interdependency of the categories is weaker with this model than the additive model because a category may still attract cherry-pickers even if it cannot attract full baskets. Because of the correlation between the utilities of the alternatives, obtaining closed-form expressions for the probabilities associated with each alternative is not possible. Although an analytical analysis of the variety competition between retailers or including pricing does not seem possible, it is possible to show that the best response of a retailer with CM provides less variety than OPT. The loss due to CM, however, may be less severe than it is with the multiplicative and additive models, depending on the value of η . If $\eta = 0$, then there is no interaction between categories and this model reduces to a case where there are no basket shoppers. If η is very large, no customers will cherry-pick, and this model reduces to the additive model in Kök (2003). Fox and Hoch (2005) report that the percentage of shopping trips where consumers cherry-pick is not very high: While

Table 8 Price and Variety Competition. Comparison of Equilibria with CM or OPT. Two Symmetric Categories

λ_j	λ_b	Z_j	c_j	A^{CM-CM}	$A^{OPT-OPT}$	ρ^{*CM-CM}	$\rho^{*OPT-OPT}$	n^{*CM-CM}	$n^{*OPT-OPT}$	π^{CM-CM}	$\pi^{OPT-OPT}$
100	100	10	0.5	21.2	32.0	1.68	1.35	113.6	123.9	77.2	43.8
100	100	10	1	9.9	14.6	1.50	1.25	44.3	51.1	44.0	25.9
100	100	10	2	3.5	5.3	1.26	1.12	12.5	16.1	13.1	7.6
100	100	20	0.5	19.8	29.2	1.50	1.25	88.5	102.2	44.0	25.9
100	100	20	1	7.1	10.6	1.26	1.12	25.1	32.3	13.1	7.6
100	100	20	2	0.0	0.0			0.0	0.0	0.0	0.0
160	40	10	0.5	26.5	30.8	1.73	1.60	148.7	152.5	108.0	91.4
160	40	10	1	13.2	15.0	1.57	1.47	63.3	65.7	71.9	62.3
160	40	10	2	5.9	6.6	1.37	1.31	23.0	24.3	34.0	30.1
160	40	20	0.5	26.4	30.1	1.57	1.47	126.5	131.3	71.9	62.3
160	40	20	1	11.7	13.1	1.37	1.31	46.1	48.6	34.0	30.1
160	40	20	2	3.2	3.5	1.14	1.12	10.0	10.8	5.5	5.0

the top-decile cherry-pick about 32% of the time, the mean and median are only 7.2% and 4.2%. This may be considered as partial support for the validity of models in which cherry-picking is not allowed.

7. Conclusion

We study the assortment planning problem with two competing retailers, each carrying multiple categories. Basket shopping consumers, i.e., consumers who desire to make a purchase in multiple categories, choose between two retailers and a no-purchase option. We investigate the retailers' assortment decisions across categories with centralized and decentralized management regimes. We find that decentralized assortment planning, as in CM, where category managers are responsible for their own category's profit, is likely to lead to lower variety, higher prices, and significantly lower profits than optimal. However, a centralized optimal solution is almost surely not implementable in practice due to the complexity of the required data estimation and optimization. Therefore, we propose a decentralized regime, like CM, but instead of evaluating each category manager's accounting profit, we measure their basket profits, where basket profits can be estimated using point-of-sale data. We find that our basket profit approach provides near-optimal solutions for a retailer. Hence, although the presence of basket shopping consumers is known to create significant analytical complications for the assortment planning problem, a robust and simple analytical solution exists.

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Appendix A. Example of a Replenishment System with Convex Operating Cost

Consider a single category with x products and demand $\lambda(x) = A(x)(A(x) + A(z))^{-1}$, where $A'(x) > 0$ and $A''(x) \leq 0$. Item i 's demand rate is $\lambda_i(x)$, where $\sum_i \lambda_i(x) = \lambda(x)$. Total shelf-space capacity is S and assume that each item's maximum inventory level is proportional to its demand rate, $S_i = S\lambda_i(x)/\lambda(x)$. There are commercial algorithms that allocate shelf space in this way (see Bultez and Naert 1988 for references). In a continuous review inventory system with zero lead time, an order is placed whenever the inventory level is zero and the average inventory level for a single product is $S_i/2$. Total average inventory is $S/2$, hence, inventory holding cost is independent of x . The number of orders per unit time for item i is $\lambda_i(x)/S_i(x) = \lambda(x)/S$, i.e., the order frequency is identical across all items. The total number of

orders in the category is $x\lambda(x)/S$, whose second derivative is

$$\begin{aligned} & \frac{2\lambda'(x) + x\lambda''(x)}{S} \\ &= \frac{A(z)}{(A(x) + A(z))^3} (2A'(x)((A(x) + A(z)) - xA'(x)) \\ & \quad + A''(x)(A(x) + A(z))) \\ & \geq \frac{A(z)}{(A(x) + A(z))^3} (2A'(x)A(z) + A''(x)(A(x) + A(z))). \end{aligned}$$

The inequality is due to the concavity of $A(x)$. $A''(x) \leq 0$ and $A(0) = 0$ implies $A(x) \geq xA'(x)$. If the x products have similar demand rates, $A(x)$ is linear. $A''(x) = 0$ and the cost function is convex in x . The cost function is still convex if $A(x)$ is not too concave. For example, if $A(x) = \ln x$, the sign of the second derivative is determined by $A(z)(-1 + 2x) - \ln x$, which is strictly positive for $z \geq e$ and $x > 1/2$.

Appendix B. Proofs

PROPOSITION 9. *The retailer's optimal assortment in category j is of the type $\{0, 1, 2, \dots, n_{rj}\}$, where $n_{rj} \leq N$.*

PROOF. The proof is by contradiction. Suppose that the optimal assortment at retailer r is $S_{rj} \subseteq S_j$ such that $l \in S_{rj}$, $i \notin S_{rj}$, and $i < l$. Let $S'_{rj} = S_{rj} \cup \{i\} \setminus \{l\}$. Because $v_{ji} \geq v_{jl}$, $e^{-p_{rj}} \sum_{m \in S'_{rj}} e^{v_{jm}} \geq e^{-p_{rj}} \sum_{m \in S_{rj}} e^{v_{jm}}$. Hence, share of retailer r for any consumer type and the category profit increase when l is replaced with i . This contradicts with the optimality of S_{rj} . \square

PROPOSITION 10. $\tilde{C}_{rj}(A_{rj})$ is a convex increasing function.

PROOF.

$$\begin{aligned} \tilde{C}'_{rj}(A_{rj}) &= C'_{rj}(G^{-1}(e^{p_{rj}} A_{rj}))(G'(G^{-1}(e^{p_{rj}} A_{rj})))^{-1} e^{p_{rj}} > 0, \\ \tilde{C}''_{rj}(A_{rj}) &= \begin{pmatrix} C''_{rj}(G^{-1}(e^{p_{rj}} A_{rj}))(G'(G^{-1}(e^{p_{rj}} A_{rj})))^{-1} \\ -C'_{rj}(G^{-1}(e^{p_{rj}} A_{rj}))(G'(G^{-1}(e^{p_{rj}} A_{rj})))^{-2} G''(G^{-1}(e^{p_{rj}} A_{rj})) \end{pmatrix} \\ & \cdot e^{2p_{rj}} \geq 0. \quad \square \end{aligned}$$

PROOF OF THEOREM 1. Define $\phi_j = A_{xj} + A_{yj} + Z_j$ and $\delta_{rj} = A_{qj} + Z_j$, where $q \neq r$

$$\frac{\partial s_{xj}}{\partial A_{xj}} = \delta_{xj} \phi_j^{-2} > 0,$$

$$\frac{\partial^2 s_{xj}}{\partial A_{xj}^2} = -2\delta_{xj} \phi_j^{-3} < 0,$$

$$\begin{aligned} \frac{\partial^2 s_{xj}}{\partial A_{yj} \partial A_{xj}} &= -2\delta_{xj} \phi_j^{-3} + \phi_j^{-2} = -\phi_j^{-3} (2\delta_{xj} - \phi_j) \\ &= -\phi_j^3 (Z_j + A_{yj} - A_{xj}). \end{aligned} \quad (B1)$$

(i) Take the second derivative of π_{xj} with respect to A_{xj} .

$$\frac{\partial^2 \pi_{xj}}{\partial A_{xj}^2} = (p_{xj} \lambda_j + p_{xj} \lambda_b s_{xk}) \frac{\partial^2 s_{xj}}{\partial A_{xj}^2} - \tilde{C}''_{xj}(A_{xj}) < 0. \quad (B2)$$

Therefore, category profit π_{xj} is concave in A_{xj} . Existence of an equilibrium follows from Theorem 1.2 in Fudenberg and Tirole (2000).

$$\frac{\partial^2 \pi_{xj}}{\partial A_{xj} \partial A_{xk}} = p_{xj} \lambda_b \frac{\partial s_{xj}}{\partial A_{xj}} \frac{\partial s_{xk}}{\partial A_{xk}} > 0 \quad \text{for } k \neq j \text{ and } j = 1, 2.$$

Applying the implicit function theorem to (3), we can show that the best response function A_{xj} of category j is increasing in A_{xk} of category k . The feasible action space $\{0 \leq A_{xj} \leq e^{-p_j} G(N_j), j = 1, 2\}$ is a nonempty, convex, and compact set. The payoff function π_{xj} is continuous in A_{xj} for any A_{yj} and supermodular in (A_{xj}, A_{xk}) . Supermodularity of π^j implies that it has increasing differences in the feasible set. Therefore, (CM) is a supermodular game. Then, by Topkis (1998, Theorem 4.2.1), the set of equilibrium points is a nonempty complete lattice; a greatest and a least equilibrium point exist. Because the payoff to a player is increasing in other players' strategies, i.e., $\partial \pi_{xj} / \partial A_{xk} \geq 0$ for $k \neq j$, the largest element is the Pareto best and the smallest element is the Pareto worst equilibrium. This result follows from a simple stepwise improvement argument; see Vives (1999, §2.2.3). It is easy to show that π_{xj} is continuous in A_{yj} for every $(\mathbf{p}_x, \boldsymbol{\lambda}, -A_{xy}, -\mathbf{c}_x)$ and it has increasing differences in A_{xj} and $(\mathbf{p}_x, \boldsymbol{\lambda}, -A_{xy}, -\mathbf{c}_x)$. By Topkis (1998, Theorem 4.2.2), we obtain the monotonicity results.

(ii) To prove uniqueness, we show that the Jacobian of (3) is a negative semidefinite matrix:

$$\begin{aligned} & \begin{vmatrix} \frac{\partial^2 \pi_{x1}}{\partial A_{x1}^2} & \frac{\partial^2 \pi_{xj}}{\partial A_{xj} \partial A_{xk}} \\ \frac{\partial^2 \pi_{x1}}{\partial A_{x1} \partial A_{x2}} & \frac{\partial^2 \pi_{x2}}{\partial A_{x2}^2} \end{vmatrix} \\ & > \prod_j p_{xj} \lambda_j \frac{\partial^2 s_{xj}}{\partial A_{xj}^2} - \prod_{j, k \neq j} p_{xj} \lambda_b \frac{\partial s_{xj}}{\partial A_{xj}} \frac{\partial s_{xk}}{\partial A_{xk}} \\ & = p_{x1} p_{x2} \left(\lambda_1 \lambda_2 \prod_{j=1,2} 2 \delta_{xj} \phi_j^{-3} - \lambda_b^2 \prod_{j=1,2} \delta_{xj}^2 \phi_j^{-4} \right) \\ & > 0 \quad \text{under (A1).} \end{aligned}$$

(iii) The proof is by contradiction. Let \mathbf{A}_x be an asymmetric equilibrium with $A_{x1} > A_{x2}$. By definition of equilibrium, \mathbf{A}_x satisfies (3). Because

$$\frac{\partial s_{x1}}{\partial A_{x1}} < \frac{\partial s_{x2}}{\partial A_{x2}}, \quad s_{x2} < s_{x1}, \quad \text{and} \quad -\tilde{c}'_{x1}(A_{x1}) < -\tilde{c}'_{x2}(A_{x2}),$$

the left-hand side of the condition for $j = 1$ is strictly less than that for $j = 2$. As a result, both conditions cannot be satisfied at the same time and \mathbf{A}_x can not be an equilibrium. \square

PROOF OF THEOREM 2. (i) Because π_{xj} are supermodular in (A_{x1}, A_{x2}) for $j = 1, 2$, so is total store profit. The monotonicity results follow from the optimization of a supermodular function.

(ii) To prove joint concavity, we show that the Jacobian of (4) is negative semidefinite. The diagonal entries in the Jacobian are negative because the profit function is concave in each decision variable. By following steps similar to the proof of Theorem 1, we can show that (A2) is a sufficient condition for the determinant of the Jacobian to be positive. \square

PROOF OF THEOREM 3. Compare the first-order conditions to the optimization problem (OPT) and the game (CM):

$$(4) - (3) = p_{xk} \lambda_b \frac{\partial s_{xj}}{\partial A_{xj}} s_{xk},$$

which is always positive if $\lambda_b > 0$. Therefore, the two sets of equations can never have the same solution. This also implies that $A_{xj}^{\text{OPT}}(A_{xk}) > A_{xj}^{\text{CM}}(A_{xk})$ for all $j \neq k$. The proof of the second part is by contradiction.

Case 1. Suppose that $\mathbf{A}_x^{\text{OPT}} < \mathbf{A}_x^{\text{CM}}$. Then,

$$\begin{aligned} \sum_j \pi_{xj}(A_{x1}^{\text{OPT}}, A_{x2}^{\text{OPT}}) & < \pi_{x1}(A_{x1}^{\text{OPT}}, A_{x2}^{\text{CM}}) + \pi_{x2}(A_{x1}^{\text{CM}}, A_{x2}^{\text{OPT}}) \\ & < \pi_{x1}(A_{x1}^{\text{CM}}, A_{x2}^{\text{CM}}) + \pi_{x2}(A_{x1}^{\text{CM}}, A_{x2}^{\text{CM}}). \end{aligned}$$

This contradicts the optimality of \mathbf{A}_x . The first inequality follows from $\partial \pi_{xj} / \partial A_{xk} \geq 0$ for $k \neq j$ and the second inequality follows from the definition of the equilibrium, i.e., $A_{xj}^{\text{CM}} = \arg \max_{A_{xj}} \pi_{xj}(A_{xj}, A_{xk})$.

Case 2. Now suppose that $A_{x2}^{\text{CM}} > A_{x2}^{\text{OPT}}$ and $A_{x1}^{\text{OPT}} > A_{x1}^{\text{CM}}$. Define the solution to (3) for j as $A_{xj}^{\text{CM}}(A_{xk})$, the best response of category j to the other category's attractiveness level in CM, and the solution to (4) for j as $A_{xj}^{\text{OPT}}(A_{xk})$, the optimal attractiveness level in j given A_{xk} . Because $A_{x2}^{\text{CM}}(A_{x1})$ is increasing and $A_{x1}^{\text{OPT}} > A_{x1}^{\text{CM}}$, we have $A_{x2}^{\text{CM}}(A_{x1}^{\text{OPT}}) > A_{x2}^{\text{CM}}(A_{x1}^{\text{CM}}) = A_{x2}^{\text{CM}}$. Because $A_{x2}^{\text{CM}}(A_{x1}^{\text{OPT}}) < A_{x2}^{\text{OPT}}(A_{x1}^{\text{OPT}}) = A_{x2}^{\text{OPT}}$, then we have $A_{x2}^{\text{OPT}} > A_{x2}^{\text{CM}}$, a contradiction. \square

PROOF OF THEOREM 4. (i) The results follow from the supermodularity of the payoff functions. We have shown that π_{rj} is supermodular in (A_{x1}, A_{x2}) in Theorem 1. From (3) and (B1), we see that π_{rj} is supermodular in $(A_{rj}, -A_{qj})$ for $r \neq q$, if $Z_j + A_{qj} > A_{rj}$, which holds because $A_{rj} < Z_j$ for any j by assumption. To show supermodularity in $(A_{rj}, -A_{qk})$ $r \neq q, j \neq k$, take the derivative of (3) with respect to A_{yk} ,

$$p_{xj} \lambda_b \frac{\partial s_{xj}}{\partial A_{xj}} \frac{\partial s_{yk}}{\partial A_{yk}} < 0.$$

For monotonicity results, it is sufficient to show the supermodularity of π_{rj} in the following pairs of variables: (A_{rj}, p_{rk}) for any j and k , $(A_{rj}, -p_{qk})$ for any j and k , $(A_{rj}, -c_{rk})$ for any j and k , (A_{rj}, c_{qk}) for any j and k .

(ii) Because the categories are symmetric, the best response of retailer r is the unique symmetric equilibrium of CM (Theorem 1). Due to symmetry, we can focus on only one component of $\mathbf{A}_r^{\text{CM}}(\mathbf{A}_q) = (A_r^{\text{CM}}(\mathbf{A}_q), A_r^{\text{CM}}(\mathbf{A}_q))$. The best response $A_r^{\text{CM}}(\mathbf{A}_q)$ is a decreasing continuous function because it is uniquely characterized by the first-order conditions in Theorem 1. Symmetric decreasing best-response functions intersect at the 45° line only once. The first-order condition is the same as (3) stated for symmetric retailers and categories. \square

PROOF OF THEOREM 5. The proof is similar to that of Theorem 4. \square

PROOF OF THEOREM 6. (i) The first part of the theorem follows from the supermodularity of CM-CM and OPT-OPT. Replacing $p_{rj} \lambda_b$ with $(p_{rj} + p_{rk}) \lambda_b$ in the first-order condition r, j of CM-CM results in the first-order conditions for OPT-OPT. Because π_{rj} are supermodular in A_{rj} and the coefficient of λ_b for all r, j , the equilibrium, is increasing in that coefficient. Therefore, the largest equilibrium of OPT-OPT is larger than that of CM-CM.

(ii) To compare the profits, we need to investigate the case where retailers jointly maximize the sum of their profits. This case provides a benchmark to compare with the

outcomes of the equilibria of the above two cases,

$$\max_{\mathbf{A}_x, \mathbf{A}_y} \sum_{r=x, y} \sum_{j=1, 2} \pi_{xj} \quad \text{s.t. } A_{rj} \geq 0, \quad (B3)$$

for $r = x, y$, and $j = 1, 2$.

The first-order conditions for optimality are as follows: For $j = 1, 2, k \neq j, r = x, y$, and $q \neq r$,

$$p_j \lambda_j \frac{\partial}{\partial A_{rj}} (s_{rj} + s_{qj}) + (p_j + p_k) \lambda_b \frac{\partial}{\partial A_{rj}} (s_{rj} s_{rk} + s_{qj} s_{qk}) - \tilde{C}'_{rj}(A_{rj}) = 0.$$

We cannot show joint concavity in the four variables, therefore we focus on symmetric solutions at the retailers, that is, $\mathbf{A}_x = \mathbf{A}_y = \mathbf{A}$. (Note that this does not imply that $A_{r1} = A_{r2}$.) A sufficient condition for concavity is (A2). First-order conditions are as follows: For $j = 1, 2, k \neq j$,

$$p_j \lambda_j Z_j \phi_j^{-2} + (p_j + p_k) \lambda_b Z_j \phi_j^{-2} A_k \phi_k^{-1} - \tilde{C}'_j(A_j) = 0.$$

Diagonal entries of the Jacobian are

$$-2p_j \lambda_j Z_j \phi_j^{-3} - 2(p_j + p_k) \lambda_b Z_j \phi_j^{-3} A_k \phi_k^{-1} - \tilde{C}''_j(A_j) A \leq 0.$$

Cross-partial are

$$(p_j + p_k) \lambda_b Z_j \phi_j^{-2} Z_k \phi_k^{-2}.$$

The Jacobian is negative semidefinite and the total profit is jointly concave if (A2) holds. If (A2) holds, then the global optimal solution of (B3) subject to $\mathbf{A}_x = \mathbf{A}_y$ is denoted $(\mathbf{A}^{\text{COL}}, \mathbf{A}^{\text{COL}})$ and is the unique solution to

$$p \lambda Z \phi^{-2} + 2p \lambda_b Z A \phi^{-3} - \tilde{C}'(A) = 0. \quad (B4)$$

Compare this with the first-order conditions in CM-CM:

$$\begin{aligned} (5) - (B4) &= p \phi^{-3} (\lambda_j \phi (\delta - Z) + \lambda_b A (\delta - 2Z)) \\ &= p \phi^{-3} A (\lambda_j \phi + \lambda_b (A - Z)) \\ &= p \phi^{-3} A (A (2\lambda_j + \lambda_b) + Z (\lambda_j - \lambda_b)) \\ &> 0 \quad \text{if and only if } A/Z \\ &> (\lambda_b - \lambda_j) / (\lambda_b + 2\lambda_j), \end{aligned}$$

which holds under (A2). (5)–(B4) > 0 implies that $A^{\text{CM-CM}} > A^{\text{COL}}$. Because $A^{\text{CM-CM}}$ is closer to the collusion outcome, the profit of each retailer with CM-CM is higher than OPT-OPT. \square

PROOF OF THEOREM 7. The proof of part (i) is due to the supermodularity of the category profits in (A_{rj}, A_{rk}) and (A_{rj}, A_{qk}) and is similar to the proofs of Theorems 1–3. Note that

$$\begin{aligned} \frac{\partial D_{rj}}{\partial A_{rj}} &= \lambda_j \delta_{rj} \phi_j^{-2} + \lambda_b \delta_{rj} \phi_j^{-2} A_{rk} \phi_k^{-1} = D_{rj} A_{rj}^{-1} \phi_j^{-1} \delta_{rj}, \\ \frac{\partial p_{rj}^*}{\partial A_{rj}} &= \frac{1}{D_{rj}} \frac{\partial D_{rj}}{\partial A_{rj}} - \frac{1}{A_{rj}} = A_{rj}^{-1} \phi_j^{-1} \delta_{rj} - A_{rj}^{-1} = -\phi_j^{-1}, \\ \frac{\partial \pi_{rj}}{\partial A_{rk}} &= \frac{\partial p_{rj}^*}{\partial A_{rk}} D_{rj} + (p_{rj}^* - 1) \frac{\partial D_{rj}}{\partial A_{rk}} = \frac{\partial D_{rj}}{\partial A_{rk}} p_{rj}^*, \\ \frac{\partial^2 \pi_{rj}}{\partial A_{rj} \partial A_{rk}} &= \frac{\partial p_{rj}^*}{\partial A_{rj}} \frac{\partial D_{rj}}{\partial A_{rk}} + \frac{\partial^2 D_{rj}}{\partial A_{rj} \partial A_{rk}} p_{rj}^* \\ &= -\phi_j^{-1} \frac{\partial D_{rj}}{\partial A_{rk}} + \frac{\partial^2 D_{rj}}{\partial A_{rj} \partial A_{rk}} p_{rj}^* \end{aligned}$$

$$\begin{aligned} &= -\phi_j^{-1} (\lambda_b A_{rj} \phi_j^{-1} \delta_{rk} \phi_k^{-2}) + (\lambda_b \delta_{rj} \phi_j^{-2} \delta_{rk} \phi_k^{-2}) p_{rj}^* \\ &= \lambda_b \phi_j^{-2} \delta_{rk} \phi_k^{-2} (\delta_{rj} p_{rj}^* - A_{rj}) \geq 0 \end{aligned}$$

because $p_{rj}^* \geq 1$ and $\delta_{rj} > Z_j > A_{rj}$ by our assumptions. Further, assume that $A_{q1} = A_{q2} = A_q$.

$$\begin{aligned} \frac{\partial^2 \pi_{rj}}{\partial A_{rj} \partial A_q} &= \frac{\partial}{\partial A_q} \left(\frac{\partial D_{rj}}{\partial A_{rj}} p_{rj}^* - \frac{D_{rj}}{A_{rj}} \right) \\ &= \frac{\partial^2 D_{rj}}{\partial A_q \partial A_{rj}} p_{rj}^* + \frac{\partial D_{rj}}{\partial A_{rj}} \frac{\partial p_{rj}^*}{\partial A_q} - \frac{1}{A_{rj}} \frac{\partial D_{rj}}{\partial A_q} \\ &= \frac{\partial^2 D_{rj}}{\partial A_q \partial A_{rj}} p_{rj}^* + \frac{\partial D_{rj}}{\partial A_q} \left(\frac{\partial D_{rj}}{\partial A_{rj}} \frac{1}{D_{rj}} - \frac{1}{A_{rj}} \right) \\ &= \frac{\partial^2 D_{rj}}{\partial A_q \partial A_{rj}} p_{rj}^* - \frac{\partial D_{rj}}{\partial A_q} A_{rj}^{-1} \phi_j^{-1} A_q \\ &= (\lambda_j \phi_j^{-3} (A_{rj} - \delta_{rj}) \\ &\quad + \lambda_b (\phi_j^{-3} (A_{rj} - \delta_{rj}) A_{rk} \phi_k^{-1} - \phi_j^{-2} \delta_{rj} A_{rk} \phi_k^{-2})) p_{rj}^* \\ &\quad + A_{rj}^{-1} \phi_j^{-1} A_q (\lambda_j \phi_j^{-2} A_{rj} \\ &\quad + \lambda_b A_{rj} A_{rk} \phi_j^{-2} \phi_k^{-2} (\phi_k + \phi_j)) < 0 \end{aligned}$$

because $\lambda_j \phi_j^{-3} ((A_{rj} - \delta_{rj}) p_{rj}^* + A_q) < 0$ and

$$\begin{aligned} &\lambda_b \phi_j^{-3} \phi_k^{-2} A_{rk} ((A_{rj} - \delta_{rj}) \phi_k - \phi_j \delta_{rj}) p_{rj}^* + A_q (\phi_k + \phi_j) \\ &= \lambda_b \phi_j^{-3} \phi_k^{-2} A_{rk} (\phi_k ((A_{rj} - \delta_{rj}) p_{rj}^* + A_q) \\ &\quad + \phi_j (-\delta_{rj} p_{rj}^* + A_q)) < 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \pi_{rk}}{\partial A_q \partial A_{rj}} &= \frac{\partial}{\partial A_q} \left(\frac{\partial D_{rk}}{\partial A_{rj}} p_{rk}^* \right) = \frac{\partial}{\partial A_q} (\lambda_b A_{rk} \phi_k^{-1} \delta_{rj} \phi_j^{-2} p_{rk}^*) \\ &= \lambda_b A_{rk} p_{rk}^* (-\phi_k^{-2} \delta_{rj} \phi_j^{-2} + \phi_k^{-1} \phi_j^{-2} - 2\phi_k^{-1} \delta_{rj} \phi_j^{-3}) \\ &\quad + \lambda_b A_{rk} \phi_k^{-1} \delta_{rj} \phi_j^{-2} \frac{\partial D_{rk}}{\partial A_q} D_{rk}^{-1} \\ &= \lambda_b A_{rk} p_{rk}^* \phi_k^{-2} \phi_j^{-3} (-\delta_{rj} \phi_j + \phi_k \phi_j - 2\delta_{rj} \phi_k) \\ &\quad + \lambda_b A_{rk} \phi_k^{-1} \delta_{rj} \phi_j^{-2} \frac{\partial D_{rk}}{\partial A_q} D_{rk}^{-1}. \end{aligned}$$

The first part is negative because $2\delta_{rj} > \phi_j$ and the second part is negative because D_{rk} is decreasing in A_q . Therefore, \mathbf{A}_r^{CM} and $\mathbf{A}_r^{\text{OPT}}$ are decreasing in A_q .

(ii) First-order conditions for CM are

$$\begin{aligned} \frac{\partial \pi_{rj}}{\partial A_{rj}} &= \left(\frac{1}{D_{rj}} \frac{\partial D_{rj}}{\partial A_{rj}} - \frac{1}{A_{rj}} \right) D_{rj} + (p_{rj}^* - 1) \frac{\partial D_{rj}}{\partial A_{rj}} \\ &= \frac{\partial D_{rj}}{\partial A_{rj}} p_{rj}^* - \frac{D_{rj}}{A_{rj}} = 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \pi_{rj}}{\partial A_{rj}^2} &= \frac{\partial^2 D_{rj}}{\partial A_{rj}^2} p_{rj}^* - \frac{\partial D_{rj}}{\partial A_{rj}} \phi_j^{-1} + \phi_j^{-2} (\lambda_j + \lambda_b A_{rk} \phi_k^{-1}) \\ &= \phi_j^{-3} (-2p_{rj}^* \delta_{rj} + A_{rj}) (\lambda_j + \lambda_b A_{rk} \phi_k^{-1}) \leq 0. \end{aligned}$$

Therefore, the category profits are concave in its own attractiveness and best-response functions are continuous. For

uniqueness, the determinant of the Jacobian is positive if

$$\begin{aligned} & \lambda_b^2 \phi_j^{-4} \delta_{rk} \delta_{rk} \phi_k^{-4} (\delta_{rj} p_{rj}^* - A_{rj}) (\delta_{rk} p_{rk}^* - A_{rk}) \\ & < \phi_j^{-3} \phi_k^{-3} (2p_{rj}^* \delta_{rj} - A_{rj}) (2p_{rk}^* \delta_{rk} - A_{rk}) \\ & \cdot (\lambda_j + \lambda_b A_{rk} \phi_k^{-1}) (\lambda_k + \lambda_b A_{rj} \phi_j^{-1}) \lambda_b^2 \leq \lambda_j \lambda_k. \end{aligned}$$

(iii) The first-order conditions for P:CM-CM and P:OPT-OPT are, respectively, the following:

$$0 = \frac{\partial p_{rj}^*}{\partial A_{rj}} D_{rj} + (p_{rj}^* - 1) \frac{\partial D_{rj}}{\partial A_{rj}} = p_{rj}^* \frac{\partial D_{rj}}{\partial A_{rj}} - \frac{D_{rj}}{A} \quad \text{for all } r, j, \quad (B5)$$

$$0 = p_{rj}^* \frac{\partial D_{rj}}{\partial A_{rj}} - \frac{D_{rj}}{A} + p_{rk}^* \frac{\partial D_{rk}}{\partial A_{rj}}, \quad k \neq j \text{ for all } r, j. \quad (B6)$$

The first-order conditions for P:OPT-OPT (B6) are always greater than that of P:CM-CM (B5).

(iv) Part (i) implies that

$$D_{rj}(\mathbf{A}_r^{\text{CM}}(\mathbf{A}_q), \mathbf{A}_q) < D_{rj}(\mathbf{A}_r^{\text{OPT}}(\mathbf{A}_q), \mathbf{A}_q),$$

which implies the result, because $n_{rj}^* = D_{rj}/c_{rj}$.

(v) The impact of an increase in A_{rj} on p_{rj}^* is negative ($\partial p_{rj}^*/\partial A_{rj} = -\phi_j^{-1} < 0$) and the impact of an increase in A_{rk} is positive ($\partial p_{rj}^*/\partial A_{rk} > 0$). However, when we consider symmetric solutions ($A_{rj}^{\text{CM}} = A_{rk}^{\text{CM}} < (A_{rj}^{\text{OPT}} = A_{rk}^{\text{OPT}})$), we have

$$\begin{aligned} \frac{\partial p_{rj}^*}{\partial A_{rj}} + \frac{\partial p_{rj}^*}{\partial A_{rk}} &= -\phi_j^{-1} + \frac{1}{D_{rj}} \frac{\partial D_{rj}}{\partial A_{rk}} \\ &= -\phi_j^{-1} + \frac{\lambda_b A_{rj} \phi_j^{-1} \delta_{rk} \phi_k^{-2}}{\lambda_j A_{rj} \phi_j^{-1} + \lambda_b A_{rj} \phi_j^{-1} A_{rk} \phi_k^{-1}}. \end{aligned}$$

Assume that $\lambda_j = \lambda_b$. The above expression is equal to

$$= -\phi_j^{-1} + \phi_k^{-1} \frac{\delta_{rk}}{\phi_k + A_{rk}} < 0,$$

and for $\lambda_j > \lambda_b$, the positive term is decreasing. Therefore, for $\lambda_j \geq \lambda_b$, $\partial p_{rj}^*/\partial A < 0$, which implies that prices are higher in CM than OPT. \square

PROOF OF THEOREM 8. (i) and (ii): We showed in Theorem 7 that the category profits and retailer profits are supermodular in $(A_{rj}, -A_q)$ for any j . Therefore, CM-CM and OPT-OPT games are supermodular. (iii) Compare the first-order conditions of P:CM-CM and P:OPT-OPT Equations ((B5) and (B6)). CM and OPT comparisons follow from $\partial D_{rk}/\partial A_{rj} > 0$.

The first-order conditions for the joint profit maximization problem in this setting are the following:

$$0 = p_{rj}^* \frac{\partial D_{rj}}{\partial A_{rj}} - \frac{D_{rj}}{A} + p_{rk}^* \frac{\partial D_{rk}}{\partial A_{rj}} + p_{qj}^* \frac{\partial D_{qj}}{\partial A_{rj}} + p_{qk}^* \frac{\partial D_{qk}}{\partial A_{rj}}, \quad r \neq q, k \neq j. \quad (B7)$$

Let $\mathbf{A}_r^{\text{COL}}$ denote the solution to (B7). First-order conditions for P:CM-CM less (B7) at a symmetric equilibrium yields

$$\begin{aligned} p^* \left(\frac{\partial D_{rk}}{\partial A_{rj}} + \frac{\partial D_{qj}}{\partial A_{rj}} + \frac{\partial D_{qk}}{\partial A_{rj}} \right) &= p^* (-\lambda_j A \phi^{-2} + \lambda_b \phi^{-3} A (\delta - A - A)) \\ &= p^* A \phi^{-3} (-\lambda_j \phi + \lambda_b (Z - A)) > 0 \end{aligned}$$

if and only if $A/Z > (\lambda_b - \lambda_j)/(2\lambda_b + \lambda_j)$,

which implies that $A^{\text{CM-CM}} > A^{\text{COL}}$ when the condition is satisfied. Results on \mathbf{n} directly follow because $n_{rj} = D_{rj}/c_{rj}$ and D_{rj} increases with A .

The impact of an increase in A_{rj} on p_{rj}^* is negative ($\partial p_{rj}^*/\partial A_{rj} = -\phi_j^{-1}$) and the impact of an increase in A_{rk} is positive ($\partial p_{rj}^*/\partial A_{rk} > 0$). However, when we consider symmetric solutions ($A_{rj}^{\text{CM-CM}} = A_{rk}^{\text{CM-CM}} < (A_{rj}^{\text{OPT-OPT}} = A_{rk}^{\text{OPT-OPT}})$ and $\lambda_j = 0$, then

$$\begin{aligned} \frac{\partial p_{rj}^*}{\partial A_{rj}} + \frac{\partial p_{rj}^*}{\partial A_{rk}} + \frac{\partial p_{rj}^*}{\partial A_{qj}} + \frac{\partial p_{rj}^*}{\partial A_{qk}} \\ &= -\phi_j^{-1} + \frac{1}{D_{rj}} \left(\frac{\partial D_{rj}}{\partial A_{rk}} + \frac{\partial D_{rj}}{\partial A_{qj}} + \frac{\partial D_{rj}}{\partial A_{qk}} \right) \\ &= -\phi_j^{-1} + (\lambda_b A_{rj} \phi_j^{-1} \delta_{rk} \phi_k^{-2} - \lambda_j A_{rj} \phi_j^{-2} - \lambda_b A_{rj} \phi_j^{-2} A_{rk} \phi_k^{-1} \\ &\quad - \lambda_b A_{rj} \phi_j^{-1} A_{rk} \phi_k^{-2}) \\ &\quad \cdot (\lambda_j A_{rj} \phi_j^{-1} + \lambda_b A_{rj} \phi_j^{-1} A_{rk} \phi_k^{-1})^{-1} \\ &= -2\phi_j^{-1} + \frac{\lambda_b Z_k \phi_k^{-2}}{\lambda_j + \lambda_b A_{rk} \phi_k^{-1}} < -2\phi_j^{-1} + \frac{\lambda_b Z_k \phi_k^{-2}}{\lambda_j + \lambda_b A_{rk} \phi_k^{-1}} \\ &= -2\phi_j^{-1} + \phi_k^{-1} \frac{Z_k}{\phi_k + A_{rk}} < 0 \quad (\text{assuming } \lambda_j = \lambda_b). \end{aligned}$$

Similar to the proof of Theorem 7, for $\lambda_j \geq \lambda_b$, prices decrease with A . \square

References

ACNielsen. 1998. *Eighth Annual Survey of Trade Promotion Practices*. ACNielsen, Chicago, IL.

Agrawal, N., S. A. Smith. 2003. Optimal retail assortments for substitutable items purchased in sets. *Naval Res. Logist.* **50**(7) 793–822.

Anderson, S. P., A. de Palma, J. F. Thisse. 1992. *Discrete Choice Theory of Product Differentiation*. MIT Press, Cambridge, MA.

Basuroy, S., D. Nguyen. 1998. Multinomial logit market share models: Equilibrium characteristics and strategic implications. *Management Sci.* **44**(10) 1396–1408.

Basuroy, S., M. K. Mantrala, R. G. Walters. 2001. The impact of category management on retailer prices and performance: Theory and evidence. *J. Marketing* **65** 16–32.

Bayus, B. L., W. P. Putsis Jr. 1999. Product proliferation: An empirical analysis of product line determinants and market outcomes. *Marketing Sci.* **18**(2) 137–153.

Bell, D. R., J. M. Lattin. 1998. Shopping behavior and consumer preference for store price format: Why large basket shoppers prefer EDLP. *Marketing Sci.* **17**(1) 66–88.

Bell, D. R., T.-H. Ho, C. S. Tang. 1998. Determining where to shop: Fixed and variable costs of shopping. *J. Marketing Res.* **35** 352–369.

Ben-Akiva, M., S. R. Lerman. 1985. *Discrete Choice Analysis: Theory and Application to Travel Demand*. MIT Press, Cambridge, MA.

Boatwright, P., J. C. Nunes. 2001. Reducing assortment: An attribute-based approach. *J. Marketing* **65**(3) 50–63.

Bodapati, A. V., V. Srinivasan. 1999. The impact of out-of-store advertising on store sales. Working paper, Northwestern University, Evanston, IL.

Broniarczyk, S. M., W. D. Hoyer, L. McAlister. 1998. Consumers' perception of the assortment offered in a grocery category: The impact of item reduction. *J. Marketing Res.* **35** 166–176.

Bultez, A., P. Naert. 1988. SHARP: Shelf allocation for retailers profit. *Marketing Sci.* **7**(3) 211–231.

- Cachon, G. P., C. Terwiesch, Y. Xu. 2005. Retail assortment planning in the presence of consumer search. *Manufacturing Service Oper. Management* 7(4) 330–346.
- Chen, Y., J. D. Hess, R. T. Wilcox, Z. J. Zhang. 1999. Accounting profits versus marketing profits: A relevant metric for category management. *Marketing Sci.* 18(3) 208–229.
- Chong, J. K., T.-H. Ho, C. S. Tang. 2001. A modeling framework for category assortment planning. *Manufacturing Service Oper. Management* 3(3) 191–210.
- Dhar, S. K., S. J. Hoch, N. Kumar. 2001. Effective category management depends on the role of the category. *J. Retailing* 77(2) 165–184.
- Draganska, M., D. Jain. 2005. Product line length as a competitive tool. *J. Econom. Management Strategy* 14(1) 1–28.
- Dreze, X., S. J. Hoch, M. E. Purk. 1994. Shelf management and space elasticity. *J. Retailing* 70 301–326.
- Fox, E. J., S. J. Hoch. 2005. Cherry-picking. *J. Marketing* 69(1) 46–62.
- Fudenberg, D., J. Tirole. 2000. *Game Theory*. MIT Press, Cambridge, MA.
- Gruca, T. S., D. Sudharshan. 1991. Equilibrium characteristics of multinomial logit market share models. *J. Marketing Res.* 28(11) 480–482.
- Hoch, S. J., E. T. Bradlow, B. Wansink. 1999. The variety of an assortment. *Marketing Sci.* 18(4) 527–546.
- Hopp, W. J., X. Xu. 2005. Product line selection and pricing with modularity in design. *Manufacturing Service Oper. Management* 7(3) 172–187.
- Hopp, W. J., X. Xu. 2006. A static approximation for dynamic demand substitution with applications in a competitive market. Working paper, Rutgers University, New Brunswick, NJ.
- Huffman, C., B. E. Kahn. 1998. Variety for sale: Mass customization or mass confusion? *J. Retailing* 74 491–513.
- Kahn, B. E., D. R. Lehmann. 1991. Modeling choice among assortment. *J. Retailing* 67 274–275.
- Karnani, A. 1985. Strategic implications of market share attraction models. *Management Sci.* 31(5) 536–547.
- Kök, A. G. 2003. Management of product variety in retail operations. Doctoral Dissertation, The Wharton School, University of Pennsylvania, Philadelphia, PA.
- Kök, A. G., M. L. Fisher. 2004. Demand estimation and assortment optimization under substitution: Methodology and application. *Oper. Res.* Forthcoming.
- Kök, A. G., M. L. Fisher, R. Vaidyanathan. 2006. Assortment planning: Review of literature and industry practice. N. Agrawal, S. A. Smith, eds. Invited chapter to appear in *Retail Supply Chain Management*. Kluwer, Amsterdam, The Netherlands.
- Kurt Salmon Associates. 1993. Efficient consumer response: Enhancing consumer value in the grocery industry. Food Marketing Institute Report 9-526, Food Marketing Institute, Arlington, VA.
- Lal, R., C. Matutes. 1989. Price competition in multimarket duopolies. *RAND J. Econom.* 20(4) 516–537.
- Lal, R., R. C. Rao. 1997. Supermarket competition: The case of everyday low pricing. *Marketing Sci.* 16(1) 60–80.
- Manchanda, P., A. Ansari, S. Gupta. 1999. The “shopping basket:” A model for multicategory purchase incidence decisions. *Marketing Sci.* 18(2) 95–114.
- Monahan, G. E. 1987. The structure of equilibria in market share attraction models. *Management Sci.* 33(2) 228–243.
- Russell, G. J., D. R. Bell, et al. 1997. Perspectives on multiple category choice. *Marketing Lett.* 8(3) 297–305.
- Simonson, I. 1999. The effect of product assortment on buyer preferences. *J. Retailing* 75 347–370.
- Smith, S. A., N. Agrawal. 2000. Management of multi-item retail inventory systems with demand substitution. *Oper. Res.* 48(1) 50–64.
- Topkis, D. M. 1998. *Supermodularity and Complementarity*. Princeton University Press, Princeton, NJ.
- van Ryzin, G., S. Mahajan. 1999. On the relationship between inventory costs and variety benefits in retail assortments. *Management Sci.* 45(11) 1496–1509.
- Vives, X. 1999. *Oligopoly Pricing: Old Ideas and New Tools*. MIT Press, Cambridge, MA.
- Walters, R. G., S. B. MacKenzie. 1988. A structural equations analysis of the impact of price promotions on store performance. *J. Marketing Res.* 25(2) 51–63.
- Zenor, M. J. 1994. The profit benefits of category management. *J. Marketing Res.* 31(2) 202–213.