Math 107, Week 8 Questions to Practice

(1) Consider a linear system $A\mathbf{x} = \mathbf{b}$ where A is 12×17.

(a) What is the smallest dimension possible for Nul A?

(b) Suppose now that the linear system has a solution $\mathbf{x}$ for every right-hand side $\mathbf{b}$. What is the rank of A?

(2) Let $T : \mathbb{P}_4 \to \mathbb{R}^5$ be the linear transformation defined by

$$T(\mathbf{p}) \; := \; \begin{bmatrix} \mathbf{p}(-2) \\ \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \\ \mathbf{p}(2) \end{bmatrix}.$$

Show that $\text{Range}(T) = \mathbb{R}^5$. (Hint: Use the rank-nullity theorem.)

(3) Let V, W be vector spaces (over $\mathbb{R}$) with bases $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$, $\widetilde{B} = \{\widetilde{\mathbf{b}}_1, \ldots, \widetilde{\mathbf{b}}_n\}$, respectively. Furthermore, suppose $T : V \to W$ is a linear transformation.

It can be shown that the transformation $\widetilde{T} : \mathbb{R}^k \to \mathbb{R}^n$ defined by $\widetilde{T}([\mathbf{v}]_B) = [\mathbf{w}]_{\widetilde{B}}$ whenever $T(\mathbf{v}) = \mathbf{w}$ is linear. Show that

$$\dim \text{Kernel}(T) \; = \; \dim \text{Kernel}(\widetilde{T})$$

(4) Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear transformation such that

$$T(1) = 1 - x + 2x^2 + x^3, \quad T(x) = 1 + 2x - x^2 - 3x^3, \quad \text{and} \quad T(x^2) = 3 + 15x - 12x^2 + 15x^3$$

Using the result of question (3) determine $\dim \text{Kernel}(T)$.

Solve the following questions from the orange textbook by Lay, Lay and McDonald.
4.4 : 29, 31
4.5 : 10, 13, 19, 20, 21
4.6 : 1, 4, 16, 23, 24, 28, 31,33