Week 7 Questions - Solutions to Selected Questions

① Note that

* $\tilde{T}: \mathbb{R}^k \to \mathbb{R}^n$ is one to one $\Leftrightarrow$ the set of columns of standard matrix A of $\tilde{T}$ is linearly independent. ... $(*)$

* A linear transformation $T: V \to W$ is one to one $\Leftrightarrow$ $\text{Kernel}\, T = \{0\}$. ... $(**)$

First we'll show that $v \in \text{Kernel}\, T \Leftrightarrow [v]_B \in \text{Kernel}\, \tilde{T}$. ..... $(***)$

$$v \in \text{Kernel}\, T \Leftrightarrow T(v) = 0_W \text{ (the zero vector in } W)$$
$$\overset{\text{by defn of } \tilde{T}}{\Leftrightarrow} \tilde{T}([v]_B) = [0_W]_{\tilde{B}} = 0$$
$$\Leftrightarrow [v]_B \in \text{Kernel}\, \tilde{T}.$$

So T is one to one $\Leftrightarrow$ $\text{Kernel}\, T = \{0\}$ (by $(**)$)

$\Leftrightarrow$ $\text{Kernel}\, \tilde{T} = \{0\}$ (by $(***)$)

$\Leftrightarrow$ $\tilde{T}$ is one to one (by $(**)$)

$\Leftrightarrow$ the set of columns of standard matrix A of $\tilde{T}$ is linearly independent.

② First note that the standard matrix A of $\tilde{T}$ is $A = [\tilde{T}(e_1)\ \tilde{T}(e_2)\ \tilde{T}(e_3)]$ where $\tilde{T}: \mathbb{R}^3 \to \mathbb{R}^3$ is linear transformation defined by

$$\tilde{T}([p]_B) = [q]_B \text{ whenever } T(p) = q.$$

and $B = \{1, x, x^2\}$ is the standar basis for $\mathbb{P}_2$.

As $[1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[x]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $[x^2]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

$$\tilde{T}(e_1) = \tilde{T}([1]_B) = [1 + x^2]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{T}(e_2) = \tilde{T}([x]_B) = [1 + 2x - x^2]_B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\tilde{T}(e_3) = \tilde{T}([x^2]_B) = [2 - 2x + 4x^2]_B = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix}$$

Check whether the colums of A are linearly independent or not.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \xrightarrow{r_3 \to -r_1 + r_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow{r_3 \to r_2 + r_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ columns of A are not linearly independent

So T is not one to one.

~~③ Consider the transformation~~ $\cancel{T: M_{2\times2} \to \mathbb{P}_4^E}$

③ Note that

$$\mathbb{P}_4^E = \left\{ p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \in \mathbb{P}_4 \;\middle|\; \overset{\forall x \in \mathbb{R}}{p(x) = p(-x)} \right\}$$

$$= \left\{ p(x) \in \mathbb{P}_4 \;\middle|\; \cancel{a_0} + a_1x + \cancel{a_2x^2} + a_3x^3 + a_4x^4 = \cancel{a_0} - a_1x + \cancel{a_2x^2} - a_3x^3 + a_4x^4 \;\; \forall x \in \mathbb{R} \right\}$$

$$= \left\{ p(x) \in \mathbb{P}_4 \;\middle|\; 2a_1x + 2a_3x^3 = 0 \;\; \forall x \in \mathbb{R} \right\}$$

$$= \left\{ p(x) \in \mathbb{P}_4 \;\middle|\; a_1 = a_3 = 0 \right\}$$

$$= \left\{ a_0 + a_2x^2 + a_4x^4 \in \mathbb{P}_4 \;\middle|\; a_0, a_2, a_4 \in \mathbb{R} \right\}$$

Consider the linear transformation $T: \tilde{M}_{2\times2} \to \mathbb{P}_4^E$ defined by

$$T\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = a + bx^2 + cx^4.$$

* <u>T is linear</u>: For any $\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} \in \tilde{M}_{2\times2}$ and for any scalar t

(i) $T\left(\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & c_1+c_2 \end{bmatrix}\right) = (a_1+a_2) + (b_1+b_2)x^2 + (c_1+c_2)x^4$

$$= (a_1 + b_1x^2 + c_1x^4) + (a_2 + b_2x^2 + c_2x^4)$$

$$= T\left(\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix}\right)$$

(ii) $$T\left(t\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} ta_1 & tb_1 \\ tb_1 & tc_1 \end{bmatrix}\right) = ta_1 + tb_1x^2 + tc_1x^4$$

$$= t(a_1 + b_1x^2 + c_1x^4)$$

$$= tT\left(\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}\right).$$

* T is onto: For any $p(x) = a_0 + a_1x^2 + a_2x^4 \in \mathbb{P}_4^E$, the matrix $\begin{bmatrix} a_0 & a_1 \\ a_1 & a_4 \end{bmatrix} \in \tilde{M}_{2\times 2}$ satisfies $T\left(\begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix}\right) = a_0 + a_1x^2 + a_2x^4$. That means T is onto.

* T is one to one: For any $\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} \in \tilde{M}_{2\times 2}$

$$T\left(\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix}\right) \Rightarrow a_1 + b_1x^2 + c_1x^4 = a_2 + b_2x^2 + c_2x^4 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow (a_1 - a_2) + (b_1 - b_2)x^2 + (c_1 - c_2)x^4 = 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow a_1 = a_2, \quad b_1 = b_2, \quad c_1 = c_2$$

$$\Rightarrow \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix}$$

So T is one to one.

T is one to one linear transformation from $\tilde{M}_{2\times 2}$ onto $\mathbb{P}_4^E$

$\Downarrow$

T is an isomorphism