Week 6 Questions

Solutions to selected questions

(1)

(a) Let us check
$$\{1-x,\ x-x^2,\ x^2-1,\ x\}$$
is linearly independent, or not.

Suppose for some $c_1, c_2, c_3, c_4 \in \mathbb{R}$
$$c_1(1-x) + c_2(x-x^2) + c_3(x^2-1) + c_4 x = 0$$
$$\implies (c_1 - c_3) + (c_2 + c_4 - c_1)x + (c_3 - c_2)x^2 = 0$$
$$\implies c_1 = c_2 = c_3 \qquad c_4 = 0$$

They are dependent, choosing $c_1 = c_2 = c_3 = 1$ and $c_4 = 0$
$$x^2 - 1 = (-1)(1-x) + (-1)(x-x^2),$$

so
$$H = \operatorname{span}\{1-x,\ x-x^2,\ x^2-1,\ x\}$$
$$= \operatorname{span}\{1-x,\ x-x^2,\ x\}$$

Let us check $\{1-x, x-x^2, x\}$ is linearly independent or not.

$$c_1(1-x) + c_2(x-x^2) + c_3 x = 0$$

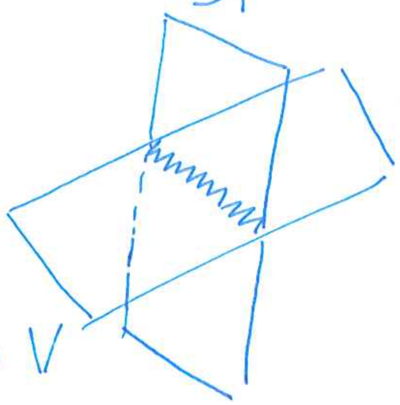
$$\Longrightarrow \quad c_1 + (c_2 + c_3 - c_1)x + (-c_2)x^2 = 0$$

$$\Longrightarrow \quad c_1 = c_2 = c_3 = 0$$

so $\{1-x, x-x^2, x\}$ is linearly independent, hence a basis for H.

(b)
$$\begin{aligned}
H &= \operatorname{span}\{1-x, x-x^2, x\} \\
&= \{c_1(1-x) + c_2(x-x^2) + c_3 x \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{\underbrace{c_1}_{d_1} + \underbrace{(c_2 + c_3 - c_1)}_{d_2}x + \underbrace{(-c_2)}_{d_3}x^2 \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{d_1 + d_2 x + d_3 x^2 \mid d_1, d_2, d_3 \in \mathbb{R}\} \\
&= \operatorname{span}\{1, x, x^2\} = \mathbb{P}_2
\end{aligned}$$

(2) (a) $S_1 \cap S_2$



vector space
and a subspace of V

closedness under $(+)$

Let $v, w \in S_1 \cap S_2$. But then

$$\begin{cases} v \in S_1 \text{ and } v \in S_2, \\ w \in S_1 \text{ and } w \in S_2 \end{cases}$$

$$\xRightarrow{\text{since } S_1, S_2 \text{ are subspaces}} v+w \in S_1 \text{ and } v+w \in S_2$$

$$\Longrightarrow v+w \in S_1 \cap S_2 .$$

closedness under $(\cdot)$

Let $v \in S_1 \cap S_2$ and $c \in \mathbb{F}$.

$$v \in S_1 \text{ and } v \in S_2$$

$$\xRightarrow{\text{since } S_1, S_2 \text{ are subspaces}} cv \in S_1 \text{ and } cv \in S_2$$

$$\Longrightarrow cv \in S_1 \cap S_2$$

(d) $S_1 \setminus S_2$ is not a subspace of V

because $0 \notin S_1 \setminus S_2$.

(3)

(a)

$$U_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$$

$$= \operatorname{span} \underbrace{\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}}_{B}$$

since $U_{2\times 2}$ can be written as a span of a set of vector (matrices which are), it is a subspace (of $M_{2\times 2}$).

Check whether B is linearly independent.

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{0}_{\substack{2\times 2 \\ 0 \text{ matrix}}}$$

$$\Longrightarrow \begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} = 0$$

$$\Longrightarrow c_1 = c_2 = c_3 = 0$$

Hence, B is linearly independent and a basis for $U_{2\times 2}$.

(b)
$$S = \{ Q \in M_{2\times 2} \mid Q^{-1} = Q^T \}$$
is not a subspace, because it is not closed under $(\cdot)$.

For instance
$$I_2^{-1} = I_2 = I_2^T$$
so $I_2 \in S$.

But
$$(2I_2)^{-1} = \frac{1}{2} I_2 \neq (2I_2)^T$$
so $2I_2 \notin S$.

(4)
(a) Recall the axioms of a vector space V (over $\mathbb{F}$) in particular $(+)$ satisfies

Basic axioms

③ $\underline{x} + \underline{y} = \underline{y} + \underline{x} \quad \forall \underline{x}, \underline{y} \in V$

④ $(\underline{x} + \underline{y}) + \underline{z} = \underline{x} + (\underline{y} + \underline{z}) \quad \forall \underline{x}, \underline{y}, \underline{z} \in V$

(5) there exists $0 \in V$ such that

$$x + 0 = x \quad \forall x \in V$$

(6) for every $x \in V$ there exists $-x \in V$ such that

$$x + (-x) = 0$$

furthermore $(\cdot)$ satisfies

(7) $(c_1 + c_2)x = c_1 x + c_2 x \quad \forall x \in V$ and $\forall c_1, c_2 \in \mathbb{F}$

(8) $c(x + y) = cx + cy \quad \forall x, y \in V$ and $\forall c \in \mathbb{F}$

(9) $c_1(c_2 x) = (c_1 c_2) x \quad \forall x \in V$ and $\forall c_1, c_2 \in \mathbb{F}$

(10) $1 \cdot x = x \quad \forall x \in V$

suppose $w \in V$ is such that

$$v + w = 0$$

(would like to show $w = -v$)

$$\xRightarrow{\text{Axiom 3}} w + v = 0$$

$$\Longrightarrow (w + v) + (-v) = 0 + (-v)$$

$$\xRightarrow{\text{Axiom 4\&3}} w + (v + (-v)) = (-v) + 0$$

$$\xRightarrow{\text{Axiom 6}} w + 0 = (-v) + 0$$

$$\xRightarrow{\text{Axiom 5}} w = -v$$

(c) This proof uses part (b).

$$(\text{Axiom } 6)\quad c\,0 = c\,(x + (-x))$$

$$\xRightarrow{\text{part (b)}}\quad c\,0 = c\,(x + (-1)\,x)$$

$$\xRightarrow{\text{Axiom } 8}\quad c\,0 = c\,x + c\,((-1)\,x)$$

$$\xRightarrow{\text{Axiom } 9}\quad c\,0 = c\,x + (-c)\,x$$

$$\xRightarrow{\text{Axiom } 7}\quad c\,0 = (c + (-c))\,x$$

$$\Longrightarrow\quad c\,0 = 0 \cdot x$$

$$\xRightarrow{0 \cdot x = 0 \ \forall x \in V \text{ is proven in class}}\quad c\,0 = 0$$