Solutions to Selected Week 13 Questions

MATH 167
SPRING 2017

Q2 Recall that if x_* is such that

$$Ax_* = \text{proj}_{Col(A)} b$$

equivalently if x_* satisfies

$$A^T A x_* = A^T b,$$

(called the normal equation)

then

$$(+) \quad \|b - Ax_*\| \leq \|b - Ax\|$$

for all x.

Let us solve

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} x_* = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b}$$

$$\iff \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x_* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

for x_*. The solution

$$x_* = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$$

satisfies (+).

You can indeed check that

$$Ax_* = \begin{bmatrix} 2/3 \\ 4/3 \\ 2/3 \end{bmatrix} \perp b - Ax_* = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}.$$

Q3

(a) We would like to minimize

$$(1) \quad \sqrt{(2-\ell(1))^2 + (3-\ell(2))^2 + (5-\ell(3))^2}$$

equivalently we would like to minimize

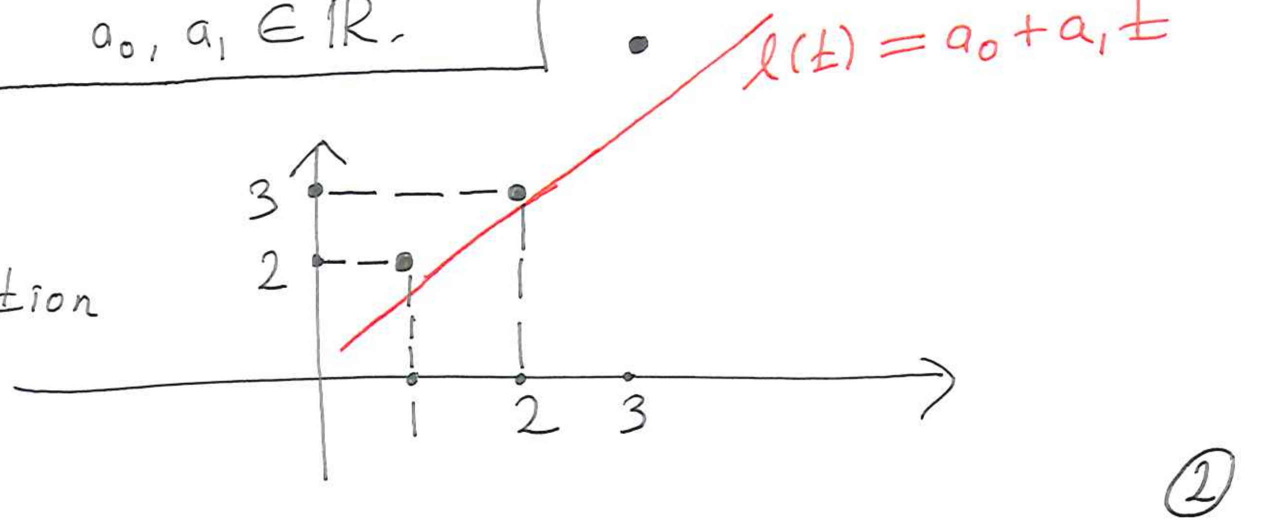
$$\left\| \begin{bmatrix} 2 - \ell(1) \\ 3 - \ell(2) \\ 5 - \ell(3) \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} a_0 + a_1 \\ a_0 + 2a_1 \\ a_0 + 3a_1 \end{bmatrix} \right\|$$

$$= \left\| \underbrace{\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}}_{b} - \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \right\|$$

over $a_0, a_1 \in \mathbb{R}$.

Illustration



(b) Solve again the normal equation.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\iff \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 10 \\ 23 \end{bmatrix}$$

$$\iff a_1 = 3/2 \qquad a_0 = 1/3$$

The line $\ell(t) = 3/2 + 1/3\, t$ minimizes the error (1).

Q4

(a) Suppose $z \in \text{Nul}\, A$. For every $\underline{y} \in \text{Row}\, A$ there exists $\alpha \in \mathbb{R}^m$ such that $\underline{y} = A^T\alpha$, so

$$\begin{aligned} z^T \underline{y} &= z^T(A^T\alpha) \\ &= (\underbrace{Az}_{0})^T \alpha = 0 \end{aligned}$$

Hence $z \perp \underline{y}$ for all $\underline{y} \in \text{Row}\, A$, that is $z \in (\text{Row}\, A)^\perp$.

Conversely, suppose $z \in (\text{Row}\, A)^\perp$. Then we have

$$z^T(A^T\alpha) = 0$$

for all $\alpha \in \mathbb{R}^m$. In particular setting $\alpha = Az$, we obtain

$$\begin{aligned} 0 = z^T(A^TAz) &= (Az)^T(Az) \\ &= \|Az\|^2 \implies Az = 0. \end{aligned}$$

Thus $z \in \text{Nul } A$ completing the proof.

Q 6

(a) Observe that

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \int_{-1}^{1} \left(\frac{1}{\sqrt{2}}\right)^2 dx = 1$$

$$\langle \cos j\pi x, \cos j\pi x \rangle = \int_{-1}^{1} \cos^2 j\pi x \, dx$$

$$\underset{\left(\text{using } \cos^2 x = \frac{1+\cos 2x}{2}\right)}{=} \int_{-1}^{1} \frac{1}{2} + \frac{\cos(2j\pi x)}{2} \, dx = 1 + \left.\frac{\sin(2j\pi x)}{2(2j\pi)}\right|_{-1}^{1}$$

$$= 1$$

$$\langle \sin j\pi x, \sin j\pi x \rangle = \int_{-1}^{1} \sin^2 j\pi x \, dx$$

$$\underset{\left(\text{using } \sin^2 x = \frac{1-\cos 2x}{2}\right)}{=} \int_{-1}^{1} \frac{1}{2} - \frac{\cos(2j\pi x)}{2} \, dx = 1 - \left.\frac{\sin(2j\pi x)}{2(2j\pi)}\right|_{-1}^{1}$$

$$= 1$$

Hence all functions in the set have norm equal to 1.

Now let us check the orthogonality.

$$\left\langle \frac{1}{\sqrt{2}}, \cos j\pi x \right\rangle = \int_{-1}^{1} \frac{\cos j\pi x}{\sqrt{2}}\, dx$$

$$= \frac{\sin j\pi x}{\sqrt{2}\,(j\pi)} \Bigg|_{-1}^{1} = 0$$

$$\left\langle \frac{1}{\sqrt{2}}, \sin j\pi x \right\rangle = \int_{-1}^{1} \frac{\sin j\pi x}{\sqrt{2}}\, dx$$

$$= -\frac{\cos j\pi x}{\sqrt{2}\,(j\pi)} \Bigg|_{-1}^{1} = \frac{-1}{(\sqrt{2}\,j\pi)}\left[\cos(j\pi) - \cos(-j\pi)\right]$$

$$= 0$$

For $j = 1, 2, \ldots$ and $k = 1, 2, \ldots$

$$\langle \cos j\pi x, \sin k\pi x \rangle = \int_{-1}^{1} \cos j\pi x \, \sin k\pi x \, dx$$

(using $\cos a \sin b = \sin(b+a) + \sin(b-a)$)

$$= \int_{-1}^{1} \sin[(j+k)\pi x] + \sin[(k-j)\pi x]\, dx$$

$$= -\frac{\cos[(j+k)\pi x]}{(j+k)\pi} \Bigg|_{-1}^{1} - \frac{\cos[(k-j)\pi x]}{(k-j)\pi} \Bigg|_{-1}^{1}$$

$$= -\Big\{ \cos[(\cancel{j+k})\pi] - \cancel{\cos}[-(j+k)\pi] \big/ ((j+k)\pi) +$$
$$\big(\cos[(\cancel{k-j})\pi] - \cos[-(\cancel{k-j})\pi]\big) \big/ ((k-j)\pi)\Big) \Big\}$$

$$= 0.$$

Hence the set is orthonormal.

(b) Due to orthonormality of

$$Q = \{1/\sqrt{2}\} \cup \{\cos j\pi x \mid j = 1, 2, \ldots\} \cup \{\sin j\pi x \mid j = 1, 2, \ldots\}$$

$$a_0 = \langle f(x), 1/\sqrt{2} \rangle$$

(In the expansion of $f(x)$ in terms of Q, take the inner product of both sides with $1/\sqrt{2}$)

$$= \int_{-1}^{1} |x| \, 1/\sqrt{2} \, dx = \frac{2}{\sqrt{2}} \int_0^1 x \, dx$$

$$= 1/\sqrt{2}$$

$$a_j = \langle f(x), \cos j\pi x \rangle$$

(In the expansion of $f(x)$, take the inner product of both sides with $\cos j\pi x$)

$$= \int_{-1}^{1} |x| \cos j\pi x \, dx = 2 \int_0^1 \underbrace{x}_{u} \, \underbrace{\cos j\pi x \, dx}_{dv}$$

use integration by parts

$$= 2 \left[x \frac{\sin j\pi x}{(j\pi)} \Big|_0^1 - \int_0^1 \frac{\sin j\pi x}{(j\pi)} dx \right]$$

$$= 2 \frac{\cos j\pi x}{(j\pi)^2} \Big|_0^1 = \frac{2}{(j\pi)^2} \left[\cos j\pi - 1 \right]$$

$$= \begin{cases} 0 & \text{if } j \text{ is even} \\ -4/(j\pi)^2 & \text{if } j \text{ is odd} \end{cases}$$

$$b_j = \langle f(x), \sin j\pi x \rangle$$

$$= \int_{-1}^{1} |x| \sin j\pi x \, dx = 0$$

because $|x| \sin j\pi x$ is an odd function

Fourier series for $f(x) = |x|$ on $[-1, 1]$

$$|x| = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_{2k-1} \cos((2k-1)\pi x)$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{-4}{[(2k-1)\pi]^2} \cos((2k-1)\pi x) \right)$$

$$= \frac{1}{2} - 4 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 \pi^2} \cos((2k-1)\pi x)$$

(c) Orthogonal projection of $f(x) = |x|$ onto span $\left\{ \frac{1}{\sqrt{2}}, \cos\pi x \right\}$ is given by

$$\frac{a_0}{\sqrt{2}} + a_1 \cos \pi x$$

$$= \frac{1}{2} - \frac{4}{\pi^2} \cos \pi x$$

Exercise:

verify that $\left\langle |x|, |x| - \left(\frac{1}{2} - \frac{4}{\pi^2} \cos \pi x \right) \right\rangle = 0.$

Q7. (a) We need to show

$$(*) \quad P(v+w) = P(v) + P(w) \quad \forall v, w \in \mathbb{R}^n$$

$$(**) \quad P(\alpha v) = \alpha P(v) \quad \forall v \in \mathbb{R}^n \ \forall \alpha \in \mathbb{R}$$

For $(*)$, letting $v_S := P(v)$, we have

$$(UD) \quad v = v_S + v_{S^\perp}$$

where $v_{S^\perp} \in S^\perp$. Similarly, letting $w_S := P(w)$, we have

$$w = w_S + w_{S^\perp}$$

where $w_{S^\perp} \in S^\perp$.

It follows that

$$v + w = (v_S + w_S) + (v_{S^\perp} + w_{S^\perp})$$

where $v_S + w_S \in S$ and $v_{S^\perp} + w_{S^\perp} \in S^\perp$ (since both S and $S^\perp$ are subspaces of $\mathbb{R}^n$, in particular they are closed under $+$).

Hence $v_S + w_S$ is the orthogonal projection of $v + w$ onto S, that is

$$P(v+w) = v_S + w_S = P(v) + P(w)$$

To prove (**), using again the decomposition (UD), we have

$$\alpha v = \alpha v_S + \alpha v_{S^\perp}$$

where $\alpha v_S \in S$ and $\alpha v_{S^\perp} \in S^\perp$ (since S and $S^\perp$ are closed under multiplication with scalars.). This implies that αv_S is the orthogonal projection of αv onto S, that is

$$P(\alpha v) = \alpha v_S = \alpha P(v)$$

as desired.

(b) As discussed in class the orthogonal projection of v onto S is given by

$$\begin{aligned}
\mathrm{proj}_S\, v &= \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 \\
&\quad + \dots + \langle v, u_k\rangle u_k \\
&= (u_1^T v) u_1 + (u_2^T v) u_2 \\
&\quad + \dots + (u_k^T v) u_k \\
&= u_1(u_1^T v) + \dots + u_k(u_k^T v) \\
&= \left[u_1 u_1^T + \dots + u_k u_k^T\right] v
\end{aligned}$$

the matrix multiplication is associative, that is $(AB)C = A(BC)$

Hence $M = u_1 u_1^T + \dots + u_k u_k^T$ as desired.

(c) First observe that for $j = 1, \ldots, k$

$$M(u_j u_j^T)$$

$$= [u_1 u_1^T + \ldots + u_k u_k^T] u_j u_j^T$$

$$= u_1 (\cancel{u_1^T u_j}) u_j^T + \ldots + u_j \overbrace{(u_j^T u_j)}^{1} u_j^T + \ldots + u_k (\cancel{u_k^T u_j}) u_j^T = u_j u_j^T.$$

Hence

$$M^2 = M(u_1 u_1^T) + M(u_2 u_2^T) + \ldots + M(u_k u_k^T)$$

$$= u_1 u_1^T + u_2 u_2^T + \ldots + u_k u_k^T$$

$$= M.$$