Math 107, Week 13 Questions to Practice

(1) Consider the inner product space $S_{2\times 2}$ consisting of 2×2 symmetric matrices with the inner product

$$\langle A, B \rangle \;:=\; a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}.$$

Find an orthonormal basis for the following subspace of $S_{2\times 2}$.

$$\left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \text{ s.t. } a + b + c = 0 \right\}.$$

(2) Letting

$$A \;=\; \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} \;=\; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

find the $\mathbf{x}$ such that $\|\mathbf{b} - A\mathbf{x}\|$ is as small as possible.

(3) (Line Fitting) There is no line in $\mathbb{R}^2$ of the form $y = \ell(t) = a_0 + a_1 t$ passing through the points

$$(1, 2), (2, 3), (3, 5).$$

(a) Express the problem of finding the line $\ell(t) = a_0 + a_1 t$ that minimizes the error

$$\sqrt{(2 - \ell(1))^2 + (3 - \ell(2))^2 + (5 - \ell(3))^2} \qquad (1)$$

as a least-squares problem, which concerns finding $a_0, a_1 \in \mathbb{R}$ such that

$$\left\| \mathbf{b} - A \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \right\|$$

is as small as possible. Above A must be a 3×2 matrix and $\mathbf{b} \in \mathbb{R}^3$.

(b) Find the line $\ell(t)$ that minimizes (1).

(4)

(a) Show that for an $m \times n$ matrix A, we have

$$(\text{Row } A)^{\perp} \;=\; \text{Nul } A.$$

(b) Specifically for

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 3 \\ 2 & 1 & 1 \end{bmatrix}$$

find Row A and Nul A. Verify for this particular example that $(\text{Row } A)^{\perp} \;=\; \text{Nul } A$.

(5) (Legendre Polynomials) Let us consider the vector space $\mathbb{P}_n$ with the inner product

$$\langle p, q \rangle \; := \; \int_{-1}^{1} p(t)q(t)\, dt$$

and the norm

$$\|p\| \; := \; \sqrt{\langle p, p \rangle} \; = \; \sqrt{\int_{-1}^{1} [p(t)]^2 \; dt.}$$

The monic polynomial in $\mathbb{P}_n$ with the smallest norm is called the Legendre polynomial of degree n. (Recall that a monic polynomial $p \in \mathbb{P}_n$ is of the form $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, that is the coefficient of the highest power term is one.)

Find the Legendre polynomial of degree 2.

(6) (Fourier Series) Every piecewise continuous function f on $[-1, 1]$ has a Fourier series of the form

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{j=1}^{\infty} a_j \cos(j\pi x) + \sum_{j=1}^{\infty} b_j \sin(j\pi x)$$

for some $a_0, a_1, a_2 \cdots \in \mathbb{R}$ and $b_1, b_2, \cdots \in \mathbb{R}$. Consider the vector space of integrable functions on the interval $[-1, 1]$ with the inner product

$$\langle f, g \rangle \; := \; \int_{-1}^{1} f(x)g(x)\, dx.$$

(a) Show that the set of functions

$$\left\{1/\sqrt{2}\right\} \cup \{\cos(j\pi x) \mid j = 1, 2, \dots\} \cup \{\sin(j\pi x) \mid j = 1, 2, \dots\}$$

is orthonormal with respect to the inner product above.

(b) Find the Fourier series specifically for $f(x) = |x|$.

(c) Find the orthogonal projection of $f(x) = |x|$ onto $\text{span}\{1/\sqrt{2}, \cos(x)\}$.

(7) (Orthogonal Projector) Let S be a subspace of $\mathbb{R}^n$ with the standard inner product, and let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be an orthonormal basis for S. Now consider the orthogonal projection transformation $P : \mathbb{R}^n \to S$ defined by $P(\mathbf{v}) = \text{proj}_S \, \mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^n$.

(a) Prove that P is linear.

(b) Show that the standard matrix of P is given by

$$M = \mathbf{u}_1 \mathbf{u}_1^T + \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \mathbf{u}_k \mathbf{u}_k^T,$$

that is $\text{proj}_S \, \mathbf{v} \; = \; M\mathbf{v}$ for every $\mathbf{v} \in V$. This standard matrix M is called the orthogonal projector onto S.
(Note: $M = \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \mathbf{u}_k \mathbf{u}_k^T$ can be expressed as $M = UU^T$ where $U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}$.)

(c) Prove that $M^2 = M$.

Solve the following questions from the orange textbook by Lay, Lay and McDonald.
6.1 : 1, 7, 13, 18, 23, 31
6.2 : 9, 20, 26
6.3 : 1, 7, 11, 17
6.4 : 7, 11
6.5 : 3, 5, 17, 24
6.7 : 7, 13, 25