Math 107, Week 12 Questions to Practice

(1) Let A be an $n \times n$ matrix with real entries. Suppose λ is a complex eigenvalue of A and v is an eigenvector corresponding to λ. Prove that $\{\operatorname{Re}\mathbf{v}, \operatorname{Im}\mathbf{v}\}$ is a linearly independent set in $\mathbb{R}^n$. (Hint: Use the fact that $\overline{\mathbf{v}}$ is an eigenvector corresponding to the eigenvalue $\overline{\lambda}$, in particular use the fact that $\{\mathbf{v}, \overline{\mathbf{v}}\}$ is linearly independent, since $\mathbf{v}, \overline{\mathbf{v}}$ correspond to two distinct eigenvalues.)

(2) Let A be a 2×2 matrix with a pair of complex conjugate eigenvalues $\lambda, \overline{\lambda}$, and let $\mathbf{v}$ be an eigenvector corresponding to the eigenvalue λ. Prove that

$$A \underbrace{\begin{bmatrix} \operatorname{Re}\mathbf{v} & \operatorname{Im}\mathbf{v} \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} \operatorname{Re}\mathbf{v} & \operatorname{Im}\mathbf{v} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{bmatrix}}_{R}.$$

Note: This shows that $P^{-1}AP = R$, so A is similar to R. Letting $|\lambda| = \sqrt{[\operatorname{Re}\lambda]^2 + [\operatorname{Im}\lambda]^2}$ be the modulus of λ, observe that $(1/|\lambda|)R$ is a rotation matrix by an angle of $\theta = \arccos\left[(\operatorname{Re}\lambda)/|\lambda|\right]$ in the counter clockwise direction. Hence $R\mathbf{x}$ first rotates by an angle of θ in the counter clockwise direction, then stretches or shrinks by $|\lambda|$.

Solve the following questions from the orange textbook by Lay, Lay and McDonald.
5.4 : 1, 8, 10
5.5 : 1, 3, 23, 24,