Math 107, Week 11 Questions to Practice

(1) Let

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where $\theta \in \mathbb{R}$. Show that A has an eigenvector in $\mathbb{R}^2$ for all $\theta \in \mathbb{R}$. In fact, there exists a vector $\mathbf{v}_1$ such that $A\mathbf{v}_1 = \mathbf{v}_1$.

(2) Let A be an $n \times n$ matrix. A subspace S is called an invariant subspace of the matrix A if

$$AS \subseteq S$$

where the set AS is defined by $AS := \{Ax \mid x \in S\}$.

Show that, for two eigenspaces E_{λ_1} and E_{λ_2} corresponding to two distinct eigenvalues λ_1 and λ_2 of A, their sum

$$E_{\lambda_1} + E_{\lambda_2} := \{\mathbf{v}_1 + \mathbf{v}_2 \mid \mathbf{v}_1 \in E_{\lambda_1},\ \mathbf{v}_2 \in E_{\lambda_2}\}$$

is an invariant subspace of A.

(3) Consider a 3×3 matrix A with three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$, and the sequence $\{\mathbf{q}_k\}$ defined by $\mathbf{q}_{k+1} = A\mathbf{q}_k$ for $k = 1, 2, \ldots$ and for a given $\mathbf{q}_0$. Show that

$$\lim_{k\to\infty} \mathbf{q}_k \ = \ 0$$

for all $\mathbf{q}_0$.

(4) The Fibonacci sequence $\{f_k\}$ is a sequence of integers such that $f_0 = 0$, $f_1 = 1$ and

$$f_{k+1} = f_k + f_{k-1} \quad \text{for } k \geq 1.$$

This recurrence can be expressed as

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} f_{k-1} \\ f_k \end{bmatrix} = \begin{bmatrix} f_k \\ f_{k+1} \end{bmatrix} \quad \text{for } k \geq 1.$$

Find a general formula for f_k (depending only on k) using the eigenvalues and eigenvectors of A.

Solve the following questions from the orange textbook by Lay, Lay and McDonald.
5.2 : 18, 20, 23, 25
5.3 : 4, 5, 11, 17, 21, 23, 31, 32
5.4 : 11, 15, 18, 19, 22, 23