1. If $n = 2$, then an n x n matrix with two identical rows is of the form $\begin{bmatrix} a & b \\ a & b \end{bmatrix}$ for some scalars a and b and

$$det\begin{bmatrix} a & b \\ a & b \end{bmatrix} = ab - ab = 0.$$

Assume that the determinant of an n x n matrix with two identical rows is equal to zero. Let M be an $(n+1)$ x $(n+1)$ matrix with two identical rows. Let the identical rows be i-th and j-th rows. Let $k \in \{1, 2, ..., n, n+1\} \setminus \{i, j\}$. Then

$$det(M) = M_{k,1}C_{k,1} + M_{k,2}C_{k,2} + ... + M_{k,(n+1)}C_{k,(n+1)}$$

where $M_{k,l}$ denotes the (k,l)-entry in M and $C_{k,l}$ denotes the (k,l)-cofactor of M. Notice that each $C_{k,l}$ is a minus or plus determinant of an n x n matrix with two identical rows. Therefore by assumption $C_{k,l} = 0$ for every l. Hence $det(M) = 0$.

2. First, we will prove that the determinant of an m x m invertible matrix M whose entries are the polynomials of degree two is a polynomial of degree at most $2m$. Since this matrix is invertible, it is row equivalent to a triangular matrix T whose diagonal entries are nonzero. Then $det(M) = k \cdot det(T)$ for some constant k. Since T is determined after applying some row operations to M, the entries of T are polynomials of degree at most 2. Therefore $det(T)$, being product of the diagonal entries of T, is a polynomial of degree at most $2m$. Since $det(M) = k \cdot det(T)$, we conclude that $det(M)$ is also a polynomial of degree at most $2m$. (You can also prove this statement by induction using definition of the determinant.)

We know that

$$A^{-1} = \frac{1}{det(A)}adj(A) = \frac{1}{det(A)}\begin{bmatrix} C_{11} & C_{21} & C_{31} & \dots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \dots & C_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \dots & C_{nn} \end{bmatrix}$$

where C_{ij}'s are the cofactors of A. Thus,

$$r_{ij}(s) = \frac{C_{ji}}{det(A)}$$

for every $i, j = 1, 2, ..., n$. By the argument above, $det(A)$ is a polynomial of degree at most $2n$. Moreover, $C_{i,j}$'s are minus or plus determinants of $(n-1)$ x $(n-1)$ matrices whose entries are the polynomials of degree two. Therefore $C_{i,j}$'s are polynomials of degree at most $2(n-1)$, by the argument above.

Hence, r_{ij}'s are rational functions and the degrees of the polynomials that appear in the numerators are at most $2n-2$ and in the denominators are at most $2n$.

3. Let $p(\lambda) := det(A - \lambda I_n)$ be the characteristic polynomial of A. We have

$$p(\lambda) = det(A - \lambda I_n) = det\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}-\lambda \end{bmatrix}.$$

In this polynomial, only term with λ^n is coming from the product $(a_{11}-\lambda) \cdot (a_{22}-\lambda) \cdot ... \cdot (a_{nn}-\lambda)$. Therefore $p(\lambda)$ is a polynomial of degree n with the leading coefficient $(-1)^n$.

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of A. But they are the roots of $p(\lambda)$, so

$$\begin{aligned} p(\lambda) &= (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n) \\ &= (-1)^n(-1)(\lambda_1 - \lambda)(-1)(\lambda_2 - \lambda)...(-1)(\lambda_n - \lambda) \\ &= (-1)^n(-1)^n(\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda) \\ &= (\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda). \end{aligned}$$

Finally, $det(A) = p(0) = \lambda_1\lambda_2...\lambda_n$.

The End.