Solutions - PS 1

MATH 106 - Calculus

October 8, 2018

These are the solutions of the problems marked with $(*)$ from PS 1.

§1.2

29.

$$\lim_{y\to 1}\frac{y-4\sqrt{y}+3}{y^2-1}=\lim_{y\to 1}\frac{(\sqrt{y}-1)(\sqrt{y}-3)}{(\sqrt{y}-1)(\sqrt{y}+1)(y+1)}=\lim_{y\to 1}\frac{\sqrt{y}-3}{(\sqrt{y}+1)(y+1)}=-\frac{1}{2}.$$

33.

$$\lim_{x\to 2}\frac{1}{x-2}-\frac{4}{x^2-4}=\lim_{x\to 2}\frac{1\cdot(x+2)-4}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\frac{1}{4}.$$

35.

$$\lim_{x\to 0}\frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2}=\lim_{x\to 0}\frac{2+x^2-(2-x^2)}{x^2\left(\sqrt{2+x^2}+\sqrt{2-x^2}\right)}=\lim_{x\to 0}\frac{2}{\left(\sqrt{2+x^2}+\sqrt{2-x^2}\right)}=\frac{1}{\sqrt{2}}.$$

63.

If $x\to 0^+$ then $f(x)=(x+\pi)^2$. Therefore $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}(x+\pi)^2=\pi^2$.

79.

We are given $|f(x)|\leq g(x)$ for all x. Suppose first that $\lim_{x\to a}g(x)=0$. Then $\lim_{x\to a}-g(x)=0$ as well. Then inequality means that

$$-g(x)\leq f(x)\leq g(x).$$

We can apply the Squeeze Theorem because $\lim_{x\to a}g(x)=\lim_{x\to a}-g(x)$ and we deduce that $\lim_{x\to a}f(x)=0$.

Now suppose that $\lim_{x\to a}g(x)=3$. Then $\lim_{x\to a}-g(x)=-3$. We cannot apply the Squeeze Theorem, and we cannot conclude that $\lim_{x\to a}f(x)$ exists. All we can say is that, if $\lim_{x\to a}f(x)$ exists, it should satisfy

$$-3\leq\lim_{x\to a}f(x)\leq 3.$$

§1.3

23.

$$\lim_{x\to 2}\frac{x-3}{x^2-4x+4}=\lim_{x\to 2}\frac{x-3}{(x-2)^2}=-\infty$$

since, $(x-2)^2 \geq 0$ for all x and, if x is sufficiently close to 2, we have $x-3<0$.

27.

$$\lim_{x\to\infty}\frac{x\sqrt{x+1}\left(1-\sqrt{2x+3}\right)}{7-6x+4x^2}=\lim_{x\to\infty}\frac{\sqrt{1+1/x}\left(1/\sqrt{x}-\sqrt{2+3/x}\right)}{7/x^2-6/x+4}=\frac{-\sqrt{2}}{4}.$$

33.

Let $f(x)=\frac{1}{\sqrt{x^2-2x}-x}$.

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{1}{\sqrt{x^2-2x}-x}=\lim_{x\to\infty}\frac{\sqrt{x^2-2x}+x}{x^2-2x-x^2}=\lim_{x\to\infty}\frac{-1}{2}\left(\sqrt{1+2/x}+1\right)=-1.$$

So the line $y=-1$ is a horizontal asymptote for $y=f(x)$ at $+\infty$.

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{1}{|x|\left(\sqrt{1-2/x}+1\right)}=0$$

so $y=0$ is a horizontal asymptote for $y=f(x)$ at $-\infty$.

Now $\sqrt{x^2-2x}-x=0$ if and only if $x=0$. So $dom(f)=\{x : x\neq 0 \text{ and } x^2-2x\geq 0\}=\mathbb{R}\setminus[0,2)=(-\infty,0)\cup[2,\infty)$.

$$\lim_{x\to 0^-}f(x)=+\infty$$

(since if $x\to 0^-$ then $x<0$ hence $-x>0$). So the line $x=0$ is a vertical asymptote.

§1.4

19.

For $-1<x<1$, the function $f(x)=x^2$ takes every value in $[0,1)$. So $f(x)<1$ for all $x\in(-1,1)$, and $f(x)$ can be made arbitrarily close to 1, but never reaches 1. Therefore $f(x)$ has no maximum on $(-1,1)$.

However $f(x)\geq 0$ for all $x\in(-1,1)$, and $f(0)=0$. So f reaches its minimum at 0.

27.

$f(x)=(x^2-1)/(x^2-4)$. The domain of f is $dom(f)=\{x : x^2-4\neq 0\}=\mathbb{R}\setminus\{\pm 2\}$.

	$(-\infty,-2)$	$(-2,-1)$	$(-1,1)$	$(1,2)$	$(2,\infty)$
$x-1$	-	-	-	+	+
$x+1$	-	-	+	+	+
$x-2$	-	-	-	-	+
$x+2$	-	+	+	+	+
$f(x)$	+	-	+	-	+

30.

$f(x) = x^3 - 15x + 1.$

$f(-4) = -3 < 0$, $f(-3) = 19 > 0$, $f(1) = -13 < 0$, $f(4) = 5 > 0$. Since f is a continuous function and switches sign at least 3 times on $[-4, 4]$, by the intermediate value theorem it follows that f has at least three roots on $[-4, 4]$. However, f is a polynomial of degree 3, so f cannot have more than 3 roots. Therefore $f(x) = 0$ has exactly 3 solutions in $[-4, 4]$.

32.

If $f(0) = 0$, then f has a fixed point at 0 and we are done. If $f(1) = 1$, then f has a fixed point at 1 and we are done.

Suppose that $f(0) \neq 0$ and $f(1) \neq 1$. Let $g(x) = f(x) - x$. Then g is also continuous.

By the extreme value theorem, g achieves its maximum M at some $x = p$, and its minimum m at some point $x = q$. Since $0 \leq f(x) \leq 1$ and $0 \leq x \leq 1$, we have $g(0) \geq 0$ and $g(1) \leq 0$. Thus $M \geq 0$ and $m \leq 0$. If $M = 0$, then $0 = g(p) = f(p) - p$ and therefore $f(p) = p$ and we are done. If $m = 0$, then $f(q) = q$ and we are done.

Suppose then that $g(q) = m < 0 < M = g(p)$. By the intermediate value theorem, there must be some c between p and q such that $g(c) = 0$ and therefore $f(c) = c$, DONE.

1 §1.5

27.

Let $M > 0$. Choose δ such that $0 < \delta < 1/M$. Then $1/\delta > M$. Thus:

$$1 < x < 1 + \delta \Rightarrow 0 < x - 1 < \delta \Rightarrow \frac{1}{\delta} < \frac{1}{x-1} \Rightarrow \frac{1}{x-1} > M.$$

Since M was arbitrary, we conclude that $\lim_{x \to 1^+} \frac{1}{x-1} = +\infty$.

29.

Let $\epsilon > 0$. Choose $M > 0$ such that $M^2 > 1/\epsilon^2 - 1$. Then $1/(M^2 + 1) < \epsilon$. Therefore

$$x > M \Rightarrow x^2 + 1 > M^2 + 1 \Rightarrow \frac{1}{x^2+1} < \frac{1}{M^2+1} < \epsilon.$$

Since ϵ was arbitrary, we conclude that

$$\lim_{x \to +\infty} \frac{1}{\sqrt{x^2+1}} = 0.$$

31.

Suppose

$$\lim_{x \to a} f(x) = L$$

and

$$\lim_{x \to a} f(x) = M.$$

Let $\epsilon > 0$. The first statement implies that there exists $\delta_1 > 0$ such that

$$|x - a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}.$$

The second statement implies that there exists $\delta_2 > 0$ such that

$$|x - a| < \delta_2 \Rightarrow |f(x) - M| < \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2.\}$ (the smallest of the two numbers). Then

$$|x-a| < \delta \Rightarrow |L - M| = |L - f(x) - (M - f(x))| \leq |f(x) - L| + |f(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon$$

(where we have used the triangular inequality in the second implication). Since ϵ is arbitrary, we conclude that $|L - M|$ is nonnegative and smaller than any positive number, so $L = M$.