Matrix Functions with Specified Eigenvalues

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Problem

Given

- an analytic matrix function $A : \mathbb{C} \to \mathbb{C}^{n \times n}$,

$$A(\lambda) = \sum_{j=1}^{\kappa} f_j(\lambda) A_j,$$

  where $f_j : \mathbb{C} \to \mathbb{C}$ are analytic and $A_j \in \mathbb{C}^{n \times n}$,
- a prescribed set of scalars $\mathbb{S} \subseteq \mathbb{C}$, and
- a positive integer r.

Determine a $\Delta \in \mathbb{C}^{n \times n}$ as small as possible in 2-norm such that $A + \Delta$ has at least r of its eigenvalues belonging to $\mathbb{S}$.

Inverse Eigenvalue Problem

Not to be confused with the following classical inverse eigenvalue problem (IEP).

Given

- a symmetric matrix function $A : \mathbb{R}^d \rightarrow \mathbb{R}^{n \times n}$ depending on its parameters smoothly,
- scalars $\lambda_1 \geq \cdots \geq \lambda_n$

determine the parameters $c \in \mathbb{R}^d$ such that $\lambda_j(A(c)) = \lambda_j$ for $j = 1, \ldots, n$.

Non-smooth variants of the Newton's method have been suggested for the affine case

$$A(c) := A_0 + \sum_{j=1}^{n} c_j A_j.$$

works of Friedland, Nocedal and Overton (1987), and Sun and Sun (2003).

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Inverse Eigenvalue Problem

Example (Sturm-Liouville)

Given $\{\lambda_j\}_{j=1}^{\infty}$, determine $p(x)$ such that

$$u''(x) + p(x)u(x) = \lambda_j u(x) \ \ x \in (0,\pi), \ \ u(0) = u(\pi) = 0$$

for each j.

Discrete version: Given $\{\lambda_j\}_{j=1}^{n}$, determine $p_1, \ldots, p_n$ such that

$$(u_{k+1} - 2u_k + u_{k-1})/h^2 + p_k u_k = \lambda_j u_k, \ \ u_0 = u_{n+1} = 0.$$

for $k = 1, \ldots, n$ and for each j, equivalently

$$\left(A(p) := \left(1/h^2 \cdot \mathrm{tri}(1,-2,1) + \mathrm{diag}(p) \right) \right) u = \lambda_j u$$

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Prescribing one eigenvalue of a matrix

Let $A(\lambda) := A - \lambda I$ for given $A \in \mathbb{C}^{n \times n}$, and $z \in \mathbb{C}$. Then

$$\Delta = -\sigma_n u_n v_n^* \text{ with } \|\Delta\|_2 = \sigma_n$$

is the smallest perturbation such that $z \in \Lambda(A + \Delta)$, where $\sigma_n := \sigma_n(A - zI)$ and u_n, v_n are the associated unit left and right singular vectors (Eckart-Young).

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Prescribing a multiple eigenvalue in connection with ill-conditioning of an eigenvalue; works of Ruhe (1969) and Wilkinson (1971, 1984)

Let $s = y^* x$ for a pair of unit left y and right x eigenvectors associated with an eigenvalue z of a matrix $A \in \mathbb{C}^{n \times n}$

Find a nearby $A + \Delta$ with z as a multiple eigenvalue.

$$\text{Wilkinson: } \exists \Delta \in \mathbb{C}^{n \times n} \text{ s.t. } \|\Delta\|_2 \leq \frac{\|A\|_2 s}{1 - s^2}$$

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Consider unitary H such that $Hx = e_1$, and $B = HAH^*$ with the eigenvectors $Hx = e_1$ and $w = Hy$.

$$B = \begin{bmatrix} z & b \\ 0 & B_1 \end{bmatrix}$$

Since, $w^* e_1 = y^* x = s$, we have $w^* = [s \;\; w_1^*]$ satisfying

$$\begin{bmatrix} s & w_1^* \end{bmatrix} \begin{bmatrix} z & b^* \\ 0 & B_1 \end{bmatrix} = z \begin{bmatrix} s & w_1^* \end{bmatrix}$$
$$\Longrightarrow$$
$$sb^* + w_1^* B_1 = z w_1^*$$
$$\Longrightarrow$$
$$w_1^* (B_1 + s w_1 b^* / w_1^* w_1) = z w_1^*$$

that is $B + \Delta$ with $\Delta := \mathrm{diag}(1, s w_1 b^* / w_1^* w_1)$ has z as a multiple eigenvalue, so is $A + H^* \Delta H$.

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Let $A(\lambda) := A - \lambda I$ for given $A \in \mathbb{C}^{n\times n}$, and $z \in \mathbb{C}$. Then

$$\Delta = -\sigma_{2n-1} U_{2n-1} V_{2n-1}^{+} \ \text{with} \ \|\Delta\|_2 = \sigma_{2n-1}$$

is the smallest perturbation such that $z \in \Lambda(A + \Delta)$ with algebraic multiplicity 2 or greater, where

$$\sigma_{2n-1} := \sup_{\gamma \geq 0} \sigma_{2n-1} \left(\begin{bmatrix} A - zI & \gamma I \\ 0 & A - zI \end{bmatrix} \right)$$

and $U_{2n-1}, V_{2n-1} \in \mathbb{C}^{n\times 2}$ such that $\text{vec}(U_{2n-1})$ and $\text{vec}(V_{2n-1})$ are unit left and right singular vectors associated with σ_{2n-1}.

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- Prescribing an eigenvalue of algebraic multiplicity three, Ikramov and Nazari (2004).
- Nearest matrices with a prescribed Jordan canonical form, Lippert (2010).

$$\|\Delta\|_2 \geq \sigma_{nr-r+1}\left(I_r \otimes A - B^T \otimes I_n\right)$$

  for any $B \in \mathbb{C}^{r\times r}$ whose Jordan form is contained in the prescribed Jordan form.
- Extensions to matrix polynomials, Psarrakos (2012)

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Sylvester Equations

Key to Malyshev's derivation is a rank characterization.

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z is a multiple eigenvalue of A

$$\Longleftrightarrow$$

$$\mathrm{rank}(A - zI)^2 \le n - 2$$

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$$\mathrm{rank}\left(\begin{bmatrix} A - zI & \gamma I \\ 0 & A - zI \end{bmatrix}\right) \le 2n - 2 \;\; \forall \gamma \ne 0$$

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$$\mathrm{rank}\left(I_2 \otimes A - \begin{bmatrix} z & -\gamma \\ 0 & z \end{bmatrix} \otimes I_n\right) \le 2n - 2 \;\; \forall \gamma \ne 0$$

$$\Longleftrightarrow \text{(see } \textit{identity}\text{)}$$

$$\dim\left\{V \in \mathbb{C}^{n\times 2} \mid AV - V\begin{bmatrix} z & 0 \\ -\gamma & z \end{bmatrix} = 0\right\} \ge 2 \;\; \forall \gamma \ne 0$$

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Theorem (Pencils)

Let $A(\lambda) = A - \lambda B$ with $A, B \in \mathbb{C}^{n \times n}$ be a pencil, which does not have any right singular block in its Kronecker canonical form, $z_1, \ldots, z_p \in \mathbb{C}$, and $r \in \mathbb{Z}^+$. The following are equivalent:

- *The multiplicities of $z_1, \ldots, z_p$ sum up to r or greater.*
- *There exist $\mu_1, \ldots, \mu_r \in \{z_1, \ldots, z_p\}$ such that*

$$\dim \left\{ X \;\middle|\; AX - BX \underbrace{\begin{bmatrix} \mu_1 & & & 0 \\ \gamma_{21} & \mu_2 & & 0 \\ & & \ddots & \\ \gamma_{r1} & & & \mu_r \end{bmatrix}}_{:=C(\mu,\gamma)} = 0 \right\} \geq r$$

for all $\gamma \in \mathcal{G}(\mu)$, the set consisting of γ values such that all Jordan blocks of $C(\mu, \gamma) \in \mathbb{C}^{r \times r}$ are of full size.

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Extension to matrix polynomials is via *linearizations*.

Let $A(\lambda) := \sum_{j=0}^{m} \lambda^j A_j$ be a polynomial for given $A_j \in \mathbb{C}^{n\times n}$ with the linearization

$$\mathcal{A}(\lambda) = \mathcal{A} - \lambda\mathcal{B} := \begin{bmatrix} 0 & I & & 0 \\ & & \ddots & 0 \\ & & & I \\ A_0 & & & A_{m-1} \end{bmatrix} - \lambda \begin{bmatrix} I & & & 0 \\ & \ddots & & \\ & & I & \\ 0 & & & -A_m \end{bmatrix}.$$

$$\mathcal{A}\left[X_0^T \dots X_{m-1}^T\right]^T - \mathcal{B}\left[X_0^T \dots X_{m-1}^T\right]^T C(\mu,\gamma) = 0$$

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Extension to analytic matrix functions is via interpolation.

Consider the analytic matrix function $A(\lambda) = \sum_{j=1}^{\kappa} f_j(\lambda) A_j$ and polynomials

$$p_j(\lambda) = \sum_{i=0}^{rp-1} c_{ji} \lambda^i \text{ such that } f_j^{(\ell)}(z_k) = p_j^{(\ell)}(z_k)$$

for $k = 1, \dots, p$, $\ell = 0, \dots, r-1$ and $j = 1, \dots, \kappa$.

Propositions

1. $f_j(C(\mu,\gamma)) = p_j(C(\mu,\gamma))$ for $j = 1, \dots, \kappa$ and $\forall \gamma \in \mathcal{G}(\mu)$
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Sylvester Equations

multiplicities of $z_1, \dots, z_p$ as eigenvalues of $A(\lambda)$
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Example on a polynomial $P(\lambda) := \sum_{j=0}^{m} \lambda^j A_j$

Prescribe z_1, z_2 with multiplicities summing up to 2 or greater

$$\text{rank}\left(\sum_{j=0}^{m} \begin{bmatrix} z_1 & \gamma \\ 0 & z_2 \end{bmatrix}^j \otimes A_j\right) = \text{rank}\left(\sum_{j=0}^{m} \begin{bmatrix} z_1^j & \gamma \cdot \frac{z_1^j - z_2^j}{z_1 - z_2} \\ 0 & z_2^j \end{bmatrix} \otimes A_j\right)$$

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Minimal Perturbation

Thus consider

$$\beta(\mu) := \inf \left\{ \|\Delta\|_2 \mid \text{rank} \left(\sum_{j=1}^{\kappa} f_j(C(\mu,\gamma))^T \otimes A_j + I \otimes \Delta \right) \leq nr - r \right\}$$

for a given μ and $\gamma \in \mathcal{G}(\mu)$.

Eckart-Young theorem yields

$$\beta(\mu) \geq \sigma_{nr-r+1} \left(\sum_{j=1}^{\kappa} f_j(C(\mu,\gamma))^T \otimes A_j \right)$$

for all $\gamma \in \mathcal{G}(\mu)$, or

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Consider the perturbation

$$\Delta = -\sigma_{nr-r+1} U_{nr-r+1} V_{nr-r+1}^{+}$$

where

- $\sigma_{nr-r+1} = \kappa(\mu)$ denotes the maximal value of the singular value assuming the supremum is attained at a γ_*, and
- U_{nr-r+1}, V_{nr-r+1} are such that $\mathrm{vec}(U_{nr-r+1})$ and $\mathrm{vec}(V_{nr-r+1})$ consist of a pair of associated left and right singular vectors of unit length.

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The fact that γ_* is a local extrema, under the assumption σ_{nr-r+1} is simple, leads to:

Theorem (Optimality Condition)

$$U^*_{nr-r+1} U_{nr-r+1} = V^*_{nr-r+1} V_{nr-r+1}$$

Consequently,

$$\begin{aligned}
\|\Delta\|_2 &= \sigma_{nr-r+1} \| U_{nr-r+1} V^+_{nr-r+1} \|_2 \\
&= \sigma_{nr-r+1} \max_{\|x\|_2=1} \sqrt{x^* (V^+_{nr-r+1})^* U^*_{nr-r+1} U_{nr-r+1} V^+_{nr-r+1} x} \\
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&= \sigma_{nr-r+1} \| V_{nr-r+1} V^+_{nr-r+1} \| = \sigma_{nr-r+1}
\end{aligned}$$

Minimal Perturbation

Additionally, $A + \Delta$ has the prescribed eigenvalues in μ with multiplicities summing up to r or greater.

$$\left(\sum_{j=1}^{\kappa} f_j(C(\mu,\gamma_*))^T \otimes A_j\right) \mathrm{vec}\left(V_{nr-r+1}\right) = \sigma_{nr-r+1} \mathrm{vec}\left(U_{nr-r+1}\right).$$

$$\Longleftrightarrow$$

$$\sum_{j=1}^{\kappa} A_j V_{nr-r+1} f_j(C(\mu,\gamma_*)) = \sigma_{nr-r+1} U_{nr-r+1}$$

$\Longleftrightarrow$ (see *assumption*)

$$\sum_{j=1}^{\kappa} A_j V_{nr-r+1} f_j(C(\mu,\gamma_*)) + \underbrace{-\sigma_{nr-r+1} U_{nr-r+1} V^{+}_{nr-r+1}}_{\Delta} V_{nr-r+1} = 0$$

Assumption: $\mathrm{rank}(V_{nr-r+1}) = r$

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Minimal Perturbation

Furthermore, the subspace

$$\mathcal{D} := \left\{ D \in \mathbb{C}^{r\times r} \mid C(\mu,\gamma)D - DC(\mu,\gamma) = 0 \right\}$$

is at least r dimensional, due to $C(\mu,\gamma) \in \mathcal{G}(\mu)$.

For any $D \in \mathcal{D}$ commuting with $p_j(C(\mu,\gamma)) = f_j(C(\mu,\gamma))$

$$\sum_{j=1}^{\kappa} A_j V_{nr-r+1} f_j(C(\mu,\gamma_*))D + \Delta V_{nr-r+1} D = 0$$

$$\sum_{j=1}^{\kappa} A_j (V_{nr-r+1} D) f_j(C(\mu,\gamma_*)) + \Delta (V_{nr-r+1} D) = 0,$$

that is

$$\dim \left\{ X \in \mathbb{C}^{n\times r} \mid \sum_{j=1}^{\kappa} A_j X\, f_j(C(\mu,\gamma_*)) + \Delta \cdot X = 0 \right\} \geq r.$$

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Minimal Perturbation

Theorem

Let $A(\lambda) := \sum_{j=1}^{\kappa} f_j(\lambda) A_j$ *with* $A_j \in \mathbb{C}^{n\times n}$ *be an analytic matrix function s.t.* $\mathrm{rank}(\tilde{A}_{rp-1}) = n$*,* $\mathbb{S} \subseteq \mathbb{C}$ *be given, and* $r \in \mathbb{Z}^+$*. Then*

$$\inf\{\|\Delta\|_2 \mid A + \Delta \text{ has } r \text{ eigenvalues in } \mathbb{S}\}$$

$$=$$

$$\inf_{\mu\in\mathbb{S}^r} \sup_{\gamma} \sigma_{nr-r+1} \left(\sum_{j=1}^{\kappa} f_j(C(\mu,\gamma))^T \otimes A_j \right)$$

and a minimal Δ *is given by* $\Delta_* = -\sigma_{nr-r+1} U_{nr-r+1} V_{nr-r+1}^{+}$ *- provided that the inf-sup problem is attained at a* (μ_*, γ_*) *where* $\sigma_{nr-r+1}(\cdot)$ *is simple and* $V_{nr-r+1} \in \mathbb{C}^{n\times r}$ *is full rank -, and where* σ_{nr-r+1} *denotes the optimal value of* $\sigma_{nr-r+1}(\cdot)$ *and* $V_{nr-1+1}, U_{nr-r+1} \in \mathbb{C}^{n\times r}$ *are s.t.* $\mathrm{vec}(V_{nr-r+1})$, $\mathrm{vec}(U_{nr-r+1})$ *are unit right and left singular vectors corresponding to* σ_{nr-r+1}.

Examples

Prescribe an eigenvalue z of multiplicity 3 to a matrix, i.e., $A(\lambda) := A - \lambda I \quad (f_1(\lambda) = 1, A_1 = A$ and $f_2(\lambda) = -\lambda, A_2 = I)$.

$$\sup_{\gamma} \sigma_{3n-2} \left(I_3 \otimes A - \begin{bmatrix} z & -\gamma_{21} & -\gamma_{31} \\ 0 & z & -\gamma_{32} \\ 0 & 0 & z \end{bmatrix} \otimes I_n \right) =$$

$$\sup_{\gamma} \sigma_{3n-2} \left(\begin{bmatrix} A - zI & \gamma_{21} I & \gamma_{31} I \\ 0 & A - zI & \gamma_{32} I \\ 0 & 0 & A - zI \end{bmatrix} \right)$$

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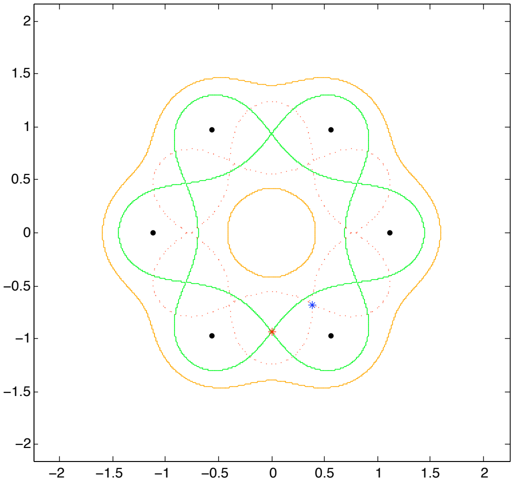
Examples



Solid orange, green - $\Lambda_\epsilon(A) = \cup_{\|\Delta\|_2 \leq \epsilon} \Lambda(A + \Delta)$ for $A \in \mathbb{C}^{6\times 6}$
Dotted red - $\Lambda_{\epsilon,2}(A) = \cup_{\|\Delta\|_2 \leq \epsilon} \Lambda_2(A + \Delta)$

Examples

Prescribe two eigenvalues z_1, z_2 with multiplicities summing up to 2 to a polynomial, i.e., $P(\lambda) := \sum_{j=0}^{m} \lambda^j A_j \quad (f_j(\lambda) = \lambda^j)$.

$$\inf_{\mu \in \{z_1, z_2\}^2} \sup_{\gamma} \sigma_{2n-1} \left(\sum_{j=0}^{m} \begin{bmatrix} \mu_1 & \gamma \\ 0 & \mu_2 \end{bmatrix}^j \otimes A_j \right) =$$

$$\inf_{\mu \in \{z_1, z_2\}^2} \sup_{\gamma} \sigma_{2n-1} \left(\begin{bmatrix} P(\mu_1) & \gamma P_{\Delta}(\mu_1, \mu_2) \\ 0 & P(\mu_2) \end{bmatrix} \right)$$

where
$P_{\Delta}(\mu_1, \mu_2) = \frac{P(\mu_1) - P(\mu_2)}{\mu_1 - \mu_2}$ if $\mu_1 \neq \mu_2$
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Examples

Prescribe two eigenvalues z_1, z_2 with multiplicities summing up to 2 to a polynomial, i.e., $P(\lambda) := \sum_{j=0}^{m} \lambda^j A_j \quad (f_j(\lambda) = \lambda^j)$.

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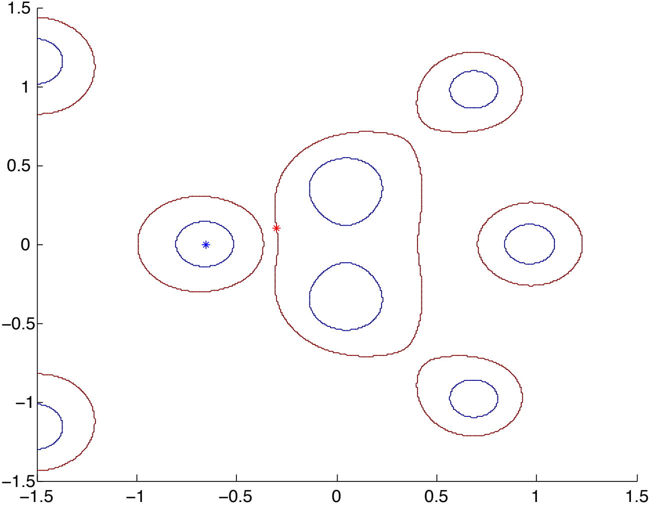

Solid brown, blue - $\Lambda_\epsilon(P) = \cup_{\|\Delta\|_2 \leq \epsilon} \Lambda(P + \Delta)$ for a 5×5 quadratic polynomial P.

Examples

Prescribe a multiple eigenvalue z to an analytic matrix function $A(\lambda) = \sum_{j=1}^{\kappa} f_j(\lambda) A_j$.

$$\sup_{\gamma} \sigma_{2n-1} \left(\sum_{j=1}^{\kappa} f_j \left(\begin{bmatrix} z & \gamma \\ 0 & z \end{bmatrix} \right) \otimes A_j \right) =$$

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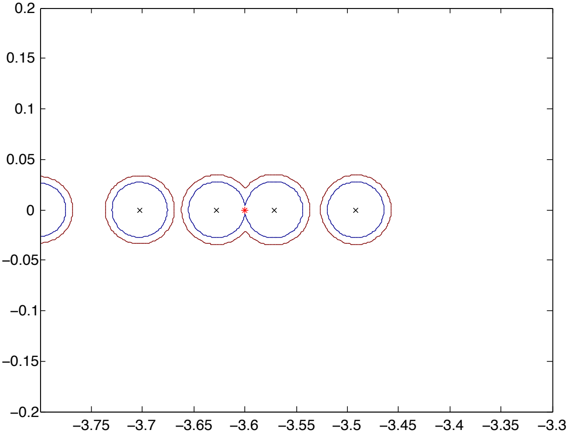

Solid brown, blue - $\Lambda_\epsilon(A) = \cup_{\|\Delta\|_2 \leq \epsilon} \Lambda(A + \Delta)$

$A(\lambda) = (e^\lambda - 1)B_1 + \lambda^2 B_2 - B_0$

$B_0 = 100 I_8$, $B_1 = [b_{jk}^{(1)} := [9 - \max(j,k)]jk]$, $B_2 = [b_{jk}^{(2)} := 9\delta_{jk} + \frac{1}{j+k}]$.

Summary and Outlook

- A computable formula for a nearest analytic matrix function with prescribed number of eigenvalues in a prescribed region.
- Results hold under a multiplicity and a full rank assumption. Future : Removal of these assumptions
- Unstructured perturbations are taken into account. Future : Structured perturbations

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Related Papers

D. Kressner, M., I. Nakic and N. Truhar. Generalized Eigenvalue Problems with Specified Eigenvalues, *IMA J. Numer. Anal., accepted subject to minor revision*

M. Karow and M. Matrix Polynomials with Specified Eigenvalues, *Math ArXiv*

M. Karow, D. Kressner and M. Nonlinear Eigenvalue Problems with Specified Eigenvalues, *in preparation*