Math 304: Numerical Methods

Instructor: Emre Mengi

Spring Semester 2010
Practice Final
Duration : 3 hours

Name ____________________

Student ID ____________________

Signature ____________________

#1	10	
#2	15	
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Total	100	

- Put your name, student ID and signature in the boxes above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.

Question 1. In each part indicate whether the statement is true or false. If the statement is false, provide a counter-example.

(i) Every $A \in \mathbb{R}^{n\times n}$ has an LU-factorization of the form $A = LU$ where $L \in \mathbb{R}^{n\times n}$ is unit lower triangular (*i.e.* a lower triangular matrix with entries on the main diagonal equal to one) and $U \in \mathbb{R}^{n\times n}$ is upper triangular.

(ii) Every $A \in \mathbb{R}^{n\times n}$ has a QR-factorization of the form $A = QR$ where $Q \in \mathbb{R}^{n\times n}$ is orthogonal and $R \in \mathbb{R}^{n\times n}$ is upper triangular.

(iii) Let $A = LU$ be an LU factorization of A where $L \in \mathbb{R}^{n\times n}$ is unit lower triangular and $U \in \mathbb{R}^{n\times n}$ is upper triangular. The ranks of A and U are the same.

(iv) Let $A = QR$ be a QR factorization of A where $Q \in \mathbb{R}^{n\times n}$ is orthogonal and $R \in \mathbb{R}^{n\times n}$ is upper triangular. Define $B := RQ$. The matrices A and B have exactly the same set of eigenvalues.

Question 2. Consider the matrices given below.

$$A_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -2 & -4 \\ -4 & -2 \end{bmatrix}$$

Note that the eigenvalues of A_1 are 4 and -2, while the eigenvalues of A_2 are -6 and 2.

(a) Suppose that the power iteration converges for both of the matrices A_1 and A_2. Which eigenvalue would you expect the power iteration to converge in each case?

(b) Suppose that the inverse iteration with the shift $\sigma = 1.5$ converges for both of the matrices A_1 and A_2. Which eigenvalue would you expect the inverse iteration with the shift $\sigma = 1.5$ to converge in each case?

(c) Suppose that the power iteration converges for both of the matrices A_1 and A_2. For which one of the matrices A_1, A_2 would you expect the power iteration to converge faster?

Question 3. This question concerns numerical integration.

(a) Suppose that you are given the quadrature nodes $x_0, x_1, \ldots, x_n \in [0, 1]$. How should you choose the weights $w_0, w_1, \ldots, w_n$ so that the degree of exactness of the quadrature formula

$$\sum_{j=0}^{n} w_j f(x_j) \approx \int_0^1 f(x)\, dx$$

is at least n?

(b) Find a quadrature formula of the form

$$w_0 f(x_0) + w_1 f(x_1) \approx \int_0^1 f(x)\, dx$$

whose degree of exactness is three. Note that you need to determine the nodes $x_0, x_1 \in [0, 1]$ as well as the weights w_0, w_1.

Question 4. The update rule for an s-step explicit multistep method is of the form

$$y_{k+s} = \phi(f, h, y_k, y_{k+1}, \dots, y_{k+s-1})$$

for the solution of the ordinary differential equation

$$y'(t) = f(t, y(t)) \; t > t_0 \;\text{ and }\; y(t_0) = y_0.$$

(a) Define the order of the s-step explicit method with the update rule as given above.

(b) Show that the order of the 2-step Adams-Bashforth method with the update rule

$$y_{k+2} = y_{k+1} + h\left(\frac{3}{2} f(t_{k+1}, y_{k+1}) - \frac{1}{2} f(t_k, y_k)\right)$$

is two.

Question 5. Consider the third-order differential equation

$$y''' + 3ty'' + ty' + 5y = 3 \; t > 0 \qquad \text{and} \qquad y(0) = 1,\; y'(0) = -1,\; y''(0) = 0$$

on the time interval $[0, 2]$.

(a) Express the third-order differential equation above as a system of first-order differential equations.

(b) Solve the resulting system of first-order differential equations using Euler's method with the step-size $h = 1$.

(c) Write down the non-linear system resulting from an application of the Trapezoidal method to the system of first order differential equations from part **(a)** with the step-size $h = 1$.

Question 6. Given the tabular data for the function $f : \mathbf{R} \to \mathbf{R}$ below.

t	1	2	3	4
$f(t)$	1	8	27	64

(a) Estimate the derivative $f'(3)$ using the forward difference formula with the step-size $h = 1$.

(b) Estimate the derivative $f'(2)$ using the backward difference formula with the step-size $h = 1$.

(c) Consider a general finite difference formula of the form

$$\mathcal{D}_\theta f(t_*) = \frac{f(t_* + h\theta) - f(t_* + h(1-\theta))}{h}$$

for the first derivative $f'(t_*)$ where $\theta \in [0,1]$. Determine all possible values of $\theta \in [0,1]$ so that

$$\mathcal{D}_\theta f(t_*) - f'(t_*) = O(h^2).$$

Question 7. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a (twice-continuously differentiable) function satisfying the Poisson equation

$$u_{xx} + u_{yy} = 9(x + y^2)$$

for $x \in (0,1)$, $y \in (0,1)$ and with the boundary conditions

$$\begin{aligned} u(x,0) = u(x,1) &= -3x \;\text{ for }\; x \in [0,1], \\ u(0,y) = u(1,y) &= -3y \;\text{ for }\; y \in [0,1]. \end{aligned}$$

(a) Set up a mesh with the step-size $h = 1/3$. Write down the linear system resulting from an application of the central difference formula for the second derivative to the Poisson equation above.

(b) Find an approximate solution for the Poisson equation above by solving the linear system from part **(a)**.