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Math 304 (Spring 2010)

Additional Questions for Midterm 2
Questions with (*) are possibly more challenging.

1. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined as $f(x) = x^p$ where $p \geq 2$ is a fixed integer.

- **(a)** Find the sequence $\{x_k\}$ generated by Newton's method. In particular write down the relation between two consecutive iterates x_{k+1} and x_k of Newton's method. (Note: Your answer must depend on p.)
- **(b)** Assume that the sequence $\{x_k\}$ in part **(a)** converges to the unique root zero. Derive the order of convergence for the sequence $\{x_k\}$.

2. To compute the reciprocal of a positive real number d without performing divisions one approach is to apply Newton's method to $f(x) = d - \frac{1}{x}$.

- **(a)** Find the sequence $\{x_k\}$ resulting from the application of Newton's method to the problem of finding the reciprocal of a positive real number d without using any divisions.
- **(b)** Derive the order of convergence for the sequence $\{x_k\}$ defined in Part **(a)** assuming $\lim_{k\to\infty} x_k = 1/d$.

3. Consider the vector-valued function $F : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$F(x) = \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ (x_1 - 1)^2 + (x_2 - 1)^2 - 1 \end{bmatrix}.$$

- **(a)** Find the linear model $L(x)$ for $F(x)$ about $x_0 = (1/2, 3/2)$ such that

$$L(x_0) = F(x_0) \ \text{ and } \ L'(x_0) = F'(x_0).$$

- **(b)** Write down the definition of a Lipschitz continuously differentiable function from $\mathbf{R}^n \to \mathbf{R}^m$. Is $F(x)$ Lipschitz continuously differentiable? Justify your answer.
- **(c)** Assume a sequence $\{x_k\}$ generated by Newton's method for $F(x)$ converges to the root $x_* = (0, 1)$. Does the sequence $\{x_k\}$ converge to $(0, 1)$ q-linearly, q-superlinearly or q-quadratically? Explain. (Note: Don't try to derive the order of convergence. Rely on a theorem discussed in class.)

4. Find the linear approximations used in Newton's method for the following functions about given points.

- **(a)**

$$f(x_1, x_2) = \begin{bmatrix} x_2 e^{x_1} + x_2^2 \\ x_2 \sin x_1 \end{bmatrix}$$

  about $(0, 1)$.
- **(b)** $f(x) = x^T x$ where $x \in \mathbb{R}^n$ about $x = [1\ 1\ \ldots\ 1]^T$ (vector of ones).
- **(c)** $f(x) = x^T A x + b^T x$ where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ about a given point $\bar{x} \in \mathbb{R}^n$.

5.(*) Let F denote any continuously differentiable function $F : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that there exists a positive constant γ such that

$$\|F'(x) - F'(y)\| \leq \gamma \|x - y\|, \text{for all } x \text{ and } y \text{ in } \mathbb{R}^n.$$

Let x_k be any point in $\mathbb{R}^n$, show that for all x,

$$\|F(x) - L_k(x)\| \leq \frac{\gamma}{2} \|x - x_k\|^2,$$

where $L(x)$ is the linear approximation about x_k.

6. (*) In this question you will establish the equivalence of the simultaneous iteration and the QR algorithm.

Algorithm 1 Simultaneous Iteration

for $k = 1, \dots$ **do**
  Compute a QR factorization $A^k = \hat{Q}_k \hat{R}_k$
  $A_k \leftarrow \hat{Q}_k^T A \hat{Q}_k$
end for

Algorithm 2 The QR Algorithm without Shifts

$A_0 \leftarrow A$
for $k = 0, 1, \dots$ **do**
  Compute a QR factorization $A_k = Q_{k+1} R_{k+1}$
  $A_{k+1} \leftarrow R_{k+1} Q_{k+1}$
end for

(a) Show by induction on k that

$$A_k = \underbrace{Q_k^T \cdots Q_2^T Q_1^T}_{\hat{Q}_k^T} A \underbrace{Q_1 Q_2 \cdots Q_k}_{\hat{Q}_k}.$$

(b) Show by induction on k that a QR factorization for A^k is given by

$$A^k = \underbrace{Q_1 Q_2 \dots Q_k}_{\hat{Q}_k} \underbrace{R_k \dots R_2 R_1}_{\hat{R}_k}.$$

(Note : Product of orthogonal matrices is another orthogonal matrix and product of upper triangular matrices is an upper triangular matrix.)

7. When the QR algorithm is applied to $A \in \mathbb{R}^{n \times n}$, the sequence of matrices generated (generically) converge to

$$\tilde{A} = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}$$

where $A_1 \in \mathbb{R}^{m \times m}, A_2 \in \mathbb{R}^{(n-m)\times(n-m)}$. Then the QR algorithm continues to iterate on A_1 and A_2. (Typically in practice when Rayleigh or Wilkinson shifts are used, the matrix A_1 is $(n-1) \times (n-1)$, so A_2 is a scalar corresponding to one of the eigenvalues of A and the QR algorithm is repeated on A_1.) This process of repeating the QR algorithm on smaller matrices is called the *deflation*.

Show that λ is an eigenvalue of A if and only if λ is eigenvalue of A_1 or A_2.

8.

(a) Apply one iteration of the QR algorithm to the matrix A provided below. (Note : $\lambda_1 = 5$ and $\lambda_2 = 1$ are the eigenvalues of A.)

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$$

(b) Show by induction that the eigenvalues of A_k are same as the eigenvalues of A for all k, where A_k is the matrix generated by the QR algorithm at the kth iteration.

9. (Numerical Mathematics, Quarteroni *et. al*)
Let $I(f) = \int_0^1 W(x)f(x)$ with $W(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = af(x_0)$. What is the maximal degree of exactness possible for $Q(f)$? Find a and x_0 such that the degree of exactness of $Q(f)$ is maximized?

10. Consider the integral

$$\int_{-0.5}^{0.5} \frac{1}{\sqrt{1-x^2}} f(x)\, dx$$

Derive the Newton-Cotes formula with three equally distant nodes $x_0 = -0.5$, $x_1 = 0$ and $x_2 = 0.5$. What is the degree of exactness for the formula you derived?

11. On May 5th the size of an ant population in the classroom SOS Z27 at Koç University is measured at various times (at various hours of the day) as listed below.

$$(\underbrace{0}_{t_0}, \underbrace{100}_{y_0}),\ (\underbrace{5}_{t_1}, \underbrace{80}_{y_1}),\ (\underbrace{10}_{t_2}, \underbrace{70}_{y_2}),\ (\underbrace{20}_{t_3}, \underbrace{40}_{y_3})$$

(a) Find the Lagrange polynomial of degree three interpolating these points. Use it to estimate the ant population at 15:30.

(b) Now suppose you would like to find the line $\ell(t) = x_1 t + x_0$ that best fits the data in the sense that

$$\sqrt{\sum_{k=0}^{3} (\ell(t_k) - y_k)^2}$$

is as small as possible. Pose this problem as a least squares problem.

12. Every rectangular matrix $A \in \mathbf{R}^{m\times n}$ with $m > n$ has the QR factorization of the form

$$\underbrace{A}_{m\times n} = \underbrace{Q}_{m\times m}\underbrace{R}_{m\times n}$$

where Q is orthogonal and R is upper triangular. Show that $\text{rank}(A) = \text{rank}(R)$, that is $\dim(\text{Col}(A)) = \dim(\text{Col}(R))$.