Instructor: Emre Mengi

Math 304 (Spring 2010)

Additional Questions for Midterm 1
Questions with (*) are possibly more challenging.

1. Given an $n \times n$ matrix A. The Matlab code provided below computes the matrix power A^n. Write down the total number of flops required by the Matlab code. You can use the big-O notation in your answer, *e.g.* if the total flop count is $3n^3 + 2n^2$, you can simply write $O(n^3)$, since asymptotically what matters is the highest order term n^3 and the term $2n^2$ becomes insignificant for large n.

```
function P = matrixpower(A)

[n,n1] = size(A);

P = A;
for j = 2:n
  P = P*A;
end

return;
```

2. Consider the linear systems

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 3 & 3 \end{bmatrix}}_{A_1} x = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}_{b_1}, \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 3 \end{bmatrix}}_{A_2} x = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}_{b_2}, \quad \text{and} \quad \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{A_3} x = \underbrace{\begin{bmatrix} 10 \\ 4 \\ 1 \end{bmatrix}}_{b_3}.$$

Which of the systems $A_1 x = b_1$, $A_2 x = b_2$ and $A_3 x = b_3$ do have unique solutions? If the system has a unique solution, solve the system by forward or back substitution.

3. For each of the following operations give the total number of flops required in terms of n. You can use the big-O notation.

- **(a)** The dot-product $x^T y$ where $x, y \in \mathbb{R}^n$
- **(b)** The matrix-vector product Ax where $x \in \mathbb{R}^n$ and A is an $n \times n$ matrix
- **(c)** The matrix-matrix product AB where A and B are $n \times n$ matrices
- **(d)** Solution of a lower triangular system $Lx = b$ for $x \in \mathbb{R}^n$ by forward substitution where L is an $n \times n$ lower-triangular matrix and $b \in \mathbb{R}^n$
- **(e)** Solution of an upper triangular system $Ux = b$ for $x \in \mathbb{R}^n$ by back substitution where U is an $n \times n$ upper triangular matrix and $b \in \mathbb{R}^n$
- **(f)** Computation of the Cholesky factorization $A = R^T R$ for a given $n \times n$ symmetric positive definite matrix A where R is upper triangular with positive diagonal entries

4. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 39 \\ 50 \\ 54 \end{bmatrix}$

(a) Find the LU factorization of A.

(b) Solve the system $A^3 x = b$ for x by exploiting the LU factorization of A from part **(a)** and without computing A^3.

5. Find an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ (using Householder reflectors) such that

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_j \\ b_{j+1} \\ b_{j+2} \\ \vdots \\ b_n \end{bmatrix} \longrightarrow \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_j \\ 0 \\ \hat{b}_{j+2} \\ \vdots \\ \hat{b}_n \end{bmatrix} = Qb,$$

that is the $j+1$th entry of the transformed vector Qb must be zero.

6. Calculate the QR factorization for the rectangular matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}$$

using the Householder reflectors. The QR factorization must be of the form

$$A = \underbrace{Q}_{4\times 4} \underbrace{R}_{4\times 2}$$

where Q is orthogonal and R is upper triangular (*i.e.* entries below r_{11} and r_{22} are zero).

7. Given a non-singular matrix $A \in \mathbf{R}^{n\times n}$ and a vector $y_0 \in \mathbf{R}^n$ Define the sequence of vectors $\{y_k\}$ for $k \geq 1$ as

$$Ay_k = y_{k-1}.$$

(a) Calculate an LU factorization for

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

by applying Gaussian elimination without pivoting.

(b) Suppose $y_0 = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$. Calculate the vectors $y_1, y_2 \in \mathbf{R}^n$ where A is as given in part **(a)** by using your LU factorization from part **(a)**, and forward and back substitutions.

(c) Write down a total flop count for the computation of $y_1, y_2, \ldots, y_n$ for a general matrix $A \in \mathbf{R}^{n \times n}$. In your total flop count provide the coefficient for the term involving highest power of n precisely. (For instance if the total flop count was $4n^2 + 8n$, an answer of the form $4n^2 + O(n)$ would be acceptable, but $O(n^2)$ would be unacceptable.)

8. Consider the matrices

$$B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}.$$

(a) Write down the characteristic polynomials for B_1, B_2 and calculate their eigenvalues.

(b) Find the eigenspace associated with each eigenvalue of B_1.

(c) Which eigenvalue and eigenvector would you expect the power iteration to converge for each of the matrices B_1 and B_2.

9.(*) Suppose $A \in \mathbf{R}^{n \times n}$ has distinct eigenvalues. Denote the eigenvalues of A by $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the associated eigenvectors by $v_1, v_2, \ldots, v_n$. Since A has distinct eigenvalues, $\lambda_j \neq \lambda_k$ for all j, k such that $j \neq k$. For simplicity assume that the eigenvalues and eigenvectors are real, that is $\lambda_j \in \mathbf{R}$, $v_j \in \mathbf{R}^n$ for $j = 1, \ldots, n$.

(a) Show that $(A - \lambda_k I)v_j = (\lambda_j - \lambda_k)v_j$.

(b) Show that the set of eigenvectors $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, that is the vector equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

with $c_1, c_2, \ldots, c_n \in \mathbf{R}$ holds only for $c_1 = c_2 = \cdots = c_n = 0$. Below an outline of a possible proof by induction is provided. It is up to you to use this outline.

(i) <u>Base case:</u> show that $\{v_1\}$ is linearly independent.

(ii) <u>Inductive case:</u> assume $\{v_1, v_2, \ldots, v_{k-1}\}$ is linearly independent for $k \geq 2$ as the inductive hypothesis. Prove that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

(iii) To prove the inductive case in **(ii)** suppose

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0.$$

Multiply both sides of the equation by $(A - \lambda_k I)$. Finally use the result from part **(a)** and the inductive hypothesis to deduce $c_1 = c_2 = \cdots = c_k = 0$.

10.(*) Consider the sequence of real numbers $\{x_k\}$ defined recursively as

$$x_{k+1} = 2x_k - 3x_k^2$$

for $k = 0, 1, 2, \ldots$ given an x_0. It can be shown that if x_0 is sufficiently close to $\frac{1}{3}$, then

$$\lim_{k \to \infty} x_k = \frac{1}{3}.$$

Show that the rate of convergence is *quadratic* when the sequence converges to $\frac{1}{3}$, that is

$$\lim_{k\to\infty} \frac{\left|x_{k+1} - \frac{1}{3}\right|}{\left|x_k - \frac{1}{3}\right|^2} = c$$

for some positive constant c.

11. Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$ write down a pseudocode to solve the system

$$A^n x = b$$

for $x \in \mathbb{R}^n$. It is essential that your pseudocode requires $O(n^3)$ flops and not $O(n^4)$ flops. (Hint: It is not a good idea to form the matrix A^n explicitly.)