Solution Set

MATH 107: Introduction to Linear Algebra

Midterm 3 - Spring 2017
Duration : 105 minutes

#1	20	
#2	25	
#3	15	
#4	20	
#5	10	
#6	10	
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NAME & LAST NAME ________________

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- Put your name, student ID and signature in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.
- Mark your section below.

SECTION 1 (EMRE MENGI TUTH 11:30-12:45) ________________
SECTION 2 (EMRE MENGI, TUTH 8:30-9:45) ________________
SECTION 3 (EMRE MENGI, MW 13:00-14:15) ________________
SECTION 4 (DOĞAN BILGE, MW 14:30-15:45) ________________

Problem 1. (20 points) Calculate the determinants of A and A^5, given that

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 0 & 3 & 2 & 2 \end{bmatrix}.$$

To calculate $\det A$, let us row-reduce A to echelon form.

$$A \xrightarrow[\substack{r_2 := r_2 - 2r_1 \\ r_3 := r_3 - 3r_1}]{} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & -6 & -8 & 2 \\ 0 & 3 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow[\substack{r_3 := r_3 - 2r_2 \\ r_4 := r_4 + r_2}]{} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & -4 & 5 \end{bmatrix}$$

$$\xrightarrow[r_4 := r_4 + r_3]{} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

Since no row-interchange is performed

$$\det A = \det U = (1)(-3)(4)(1) = -12$$

By the multiplicative property of the determinant

$$\det A^5 = (\det A)^5$$
$$= (-12)^5$$

Problem 2. This problem concerns a 3×3 matrix A with $\lambda_1 = 2$ as an eigenvalue with algebraic multiplicity 2 and $\lambda_2 = -1$ as another eigenvalue with algebraic multiplicity 1.

Suppose also that $\mathbf{v}_1 = (1, 0, 0), \mathbf{v}_2 = (1, 1, 0)$ are two eigenvectors of A corresponding to $\lambda_1 = 2$, and $\mathbf{v}_3 = (0, 1, 1)$ is an eigenvector of A corresponding to $\lambda_2 = -1$.

(a) (15 points) Is A diagonalizable? If it is diagonalizable, write down a 3×3 invertible matrix S and a 3×3 diagonal matrix D such that $S^{-1}AS = D$. If it is not diagonalizable, explain why it is not diagonalizable.

A is diagonalizable, because A has 3 linearly independent eigenvectors, that is $\{v_1, v_2, v_3\}$ is linearly independent.

Indeed we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} A \underbrace{\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}}_{S} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D}$$

λ_1 λ_1 λ_2

(b) (10 points) Calculate A^{10}.

$$A = S D S^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$A^{10} = S D^{10} S^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & (-1)^{10} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Let us calculate S^{-1}

$$\left[\underbrace{\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}}_{S} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3}\right] \xrightarrow[r_2 := r_2 - r_3]{} \left[\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}\right] \xrightarrow[r_1 := r_1 - r_2]{} \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \underbrace{\begin{matrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}}_{S^{-1}}\right]$$

Hence

$$A^{10} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & (-1)^{10} \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 2^{10} & 1-2^{10} \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3. (15 points) Consider a 3×3 matrix A with three distinct eigenvalues $\lambda_1 = 1/2, \lambda_2 = 1/4, \lambda_3 = 1/8$. Consider also the sequence $\{q_k\}$ defined by

$$q_{k+1} = Aq_k \quad \text{for } k = 1, 2, 3, \ldots$$

and for a given $q_0 \in \mathbb{R}^3$.

Show that $\lim_{k\to\infty} q_k = 0$ for all $q_0 \in \mathbb{R}^3$.

Let v_1, v_2, v_3 be the eigenvectors corresponding to $\lambda_1, \lambda_2, \lambda_3$. The set $\{v_1, v_2, v_3\}$ is linearly independancdent, because $\lambda_1, \lambda_2, \lambda_3$ are distinct. Hence, span$\{v_1, v_2, v_3\}$ is the whole space, in particular it contains q_0, that is

$$q_0 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

for some scalars $\alpha_1, \alpha_2, \alpha_3$.

Furthermore observe that

$$(*) \quad q_k = A^k q_0 = \alpha_1 (A^k v_1) + \alpha_2 (A^k v_2) + \alpha_3 (A^k v_3).$$

Also observe

$$A v_j = \lambda_j v_j$$
$$A^2 v_j = \lambda_j A v_j = \lambda_j^2 v_j$$
$$\vdots$$
$$A^k v_j = \lambda_j^k v_j$$

for $j = 1, 2, 3$. It follows from $(*)$ that

$$q_k = \alpha_1 \lambda_1^k v_1 + \alpha_2 \lambda_2^k v_2 + \alpha_3 \lambda_3^k v_3.$$

Taking the limit as $k \to \infty$ yields

$$\lim_{k\to\infty} q_k = \left(\lim_{k\to\infty} \lambda_1^k\right)\alpha_1 v_1 + \left(\lim_{k\to\infty} \lambda_2^k\right)\alpha_2 v_2 + \left(\lim_{k\to\infty} \lambda_3^k\right)\alpha_3 v_3 = 0.$$

Problem 4.

(a) (8 points) Suppose λ is a complex eigenvalue (that is $\operatorname{Im}\lambda \neq 0$) of an $n \times n$ real matrix A, and $\mathbf{v}$ is an eigenvector corresponding to λ. Show that $\overline{\lambda}$ is also an eigenvalue of A, and $\overline{\mathbf{v}}$ is an eigenvector corresponding to $\overline{\lambda}$.

$$Av = \lambda v$$

since A is real

$$\Longrightarrow A\overline{v} = \overline{A}\,\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v}$$

Hence, $\overline{\lambda}$ is also an eigenvalue of A with the corresponding eigenvector $\overline{v}$.

(b) (12 points) Find the eigenspace corresponding to each eigenvalue of

$$A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}.$$

First find the eigenvalues

$$p(\lambda) = \det(A - \lambda I_2) = \det\begin{bmatrix} 1-\lambda & -1 \\ 4 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2 + 4$$

$$= \lambda^2 - 2\lambda + 5$$

Eigenvalues (roots of $p(\lambda)$): $\lambda_1 = \dfrac{2+4i}{2} = 1+2i \qquad \lambda_2 = 1-2i$

Eigenspace corr. to λ_1

$$E_{\lambda_1} = \operatorname{Nul}(A - (1+2i)I) = \operatorname{Nul}\begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix}$$

General solution of $\begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$(-2i)x_1 + (-1)x_2 = 0$$
$$(4)x_1 + (-2i)x_2 = 0$$

(1st eqn) $\times (2i)$ = 2nd eqn. Thus discard first eqn

x_2 - free $\quad x_1 = \left(\frac{i}{2}\right)x_2$

$$E_{\lambda_1} = \left\{ \begin{bmatrix} (i/2)x_2 \\ x_2 \end{bmatrix} \,\middle|\, x_2 \in \mathbb{C} \right\} = \operatorname{span}\left\{ \begin{bmatrix} i/2 \\ 1 \end{bmatrix} \right\}$$

Eigenspace corr. to λ_2

$$E_{\lambda_2} = \operatorname{Nul}(A - (1-2i)I) = \operatorname{Nul}\begin{bmatrix} 2i & -1 \\ 4 & 2i \end{bmatrix}$$

General solution of $\begin{bmatrix} 2i & -1 \\ 4 & 2i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$(2i)x_1 - x_2 = 0$$
$$(4)x_1 + (2i)x_2 = 0$$

(1st eqn) $\times (-2i)$ = 2nd eqn

Discard first eqn

x_2 - free $\quad x_1 = -(i/2)x_2$

$$E_{\lambda_2} = \left\{ \begin{bmatrix} -(i/2)x_2 \\ x_2 \end{bmatrix} \,\middle|\, x_2 \in \mathbb{C} \right\} = \operatorname{span}\left\{ \begin{bmatrix} -i/2 \\ 1 \end{bmatrix} \right\}$$

Problem 5. (10 points) Consider a 3×3 matrix

$$A = \begin{bmatrix} q_{11}(s) & q_{12}(s) & q_{13}(s) \\ q_{21}(s) & q_{22}(s) & q_{23}(s) \\ q_{31}(s) & q_{32}(s) & q_{33}(s) \end{bmatrix}$$

that depends on a real parameter s, where $q_{jk} : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree three in s for $j, k = 1, 2, 3$. Now consider those s such that A is invertible, and for such s let

$$A^{-1} = \begin{bmatrix} r_{11}(s) & r_{12}(s) & r_{13}(s) \\ r_{21}(s) & r_{22}(s) & r_{23}(s) \\ r_{31}(s) & r_{32}(s) & r_{33}(s) \end{bmatrix}.$$

What kind of functions are $r_{jk} : \mathbb{R} \to \mathbb{R}$ for $j, k = 1, 2, 3$? For instance, are they polynomials, if yes what are their degree? Or, are they rational functions (that is can they be expressed as the ratio of two polynomials), if yes what are the degrees of the polynomials that appear in the numerators and the denominators of the rational functions? Support your answer.

Recall that

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

where

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

and $C_{ij} = (-1)^{i+j} \det A_{ij}$, A_{ij} is the matrix obtained from A by removing it's ith row and jth column.

Observe that $C_{ij} \in \mathbb{P}_6$ for $i,j = 1,2,3$ whereas $\det A \in \mathbb{P}_9$.

Hence

$$r_{ij}(s) \in \mathbb{P}_{6/9} := \left\{ \frac{p(s)}{q(s)} \;\middle|\; p \in \mathbb{P}_6 \text{ and } q \in \mathbb{P}_9 \right\}$$

for $i,j = 1,2,3$, that is each $r_{ij}(s)$ is a rational function with the polynomial in the numerator is of degree 6 at most and the polynomial in the denominator is of degree 9 at most.

Problem 6. For every $n \times n$ matrix A, there exist $n \times n$ invertible matrices V, W such that

$$WAV = D \tag{1}$$

is a diagonal matrix. By making use of this fact, in parts (a) and (b) below, either prove that the statement is true, or otherwise give a counter example proving that the statement is false.

(a) (5 points) A and D in (1) necessarily have the same eigenvalues.

FALSE

For instance, if A is invertible with all eigenvalues different than 1, for $W = A^{-1}$ and $V = I_n$ we have

$$WAV = I_n$$

but the eigenvalues of A and I_n are not the same.

More specifically choose $A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}$, $W = A^{-1} = \begin{bmatrix} 1/4 & -1/8 \\ 0 & 1/2 \end{bmatrix}$, $V = I_2$

$WAV = I_2$ but neither of the eigenvalues of A ~~are not~~ is 1.

(b) (5 points) Letting $T(\mathbf{x}) = A\mathbf{x}$, there exist two bases B and C for $\mathbb{R}^n$ such that

$$[T(\mathbf{x})]_C = D[\mathbf{x}]_B \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

where A and D are matrices as in (1) (in particular D is a diagonal matrix).

TRUE

Let us start from (1), that is

$$WAV = D$$

for some invertible W, V and diagonal D implying

$$A = W^{-1} D V^{-1}$$

Hence, defining $U := W^{-1}$,

$$T(x) = Ax = (W^{-1} D V^{-1}) x$$

$$\Longrightarrow W\,T(x) = D\,(V^{-1}x) \Longrightarrow (+)\; U^{-1} T(x) = D\,(V^{-1}x)$$

Letting $U = [u_1 \cdots u_n]$ and $V = [v_1 \cdots v_n]$, also $B = \{v_1, \dots, v_n\}$ and $C = \{u_1, \dots, u_n\}$, we have

$$U^{-1} T(x) = [T(x)]_C \text{ and } V^{-1} x = [x]_B .$$

Thus (+) can be written as $[T(x)]_C = D[x]_B$ as desired.