MATH 107: Introduction to Linear Algebra

Midterm 3 - Spring 2017

Duration : 105 minutes

#1	20	
#2	25	
#3	15	
#4	20	
#5	10	
#6	10	
Σ	100	

NAME & LAST NAME ________________

STUDENT ID ________________

SIGNATURE ________________

- Put your name, student ID and signature in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.
- Mark your section below.

  SECTION 1 (EMRE MENGI TUTH 11:30-12:45) ________________

  SECTION 2 (EMRE MENGI, TUTH 8:30-9:45) ________________

  SECTION 3 (EMRE MENGI, MW 13:00-14:15) ________________

  SECTION 4 (DOĞAN BILGE, MW 14:30-15:45) ________________

Problem 1. (20 points) Calculate the determinants of A and A^5, given that

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 0 & 3 & 2 & 2 \end{bmatrix}.$$

Problem 2. This problem concerns a 3×3 matrix A with $\lambda_1 = 2$ as an eigenvalue with algebraic multiplicity 2 and $\lambda_2 = -1$ as another eigenvalue with algebraic multiplicity 1.

Suppose also that $\mathbf{v}_1 = (1,0,0), \mathbf{v}_2 = (1,1,0)$ are two eigenvectors of A corresponding to $\lambda_1 = 2$, and $\mathbf{v}_3 = (0,1,1)$ is an eigenvector of A corresponding to $\lambda_2 = -1$.

(a) (15 points) Is A diagonalizable? If it is diagonalizable, write down a 3×3 invertible matrix S and a 3×3 diagonal matrix D such that $S^{-1}AS = D$. If it is not diagonalizable, explain why it is not diagonalizable.

(b) (10 points) Calculate A^{10}.

Problem 3. (15 points) Consider a 3×3 matrix A with three distinct eigenvalues $\lambda_1 = 1/2, \lambda_2 = 1/4, \lambda_3 = 1/8$. Consider also the sequence $\{\mathbf{q}_k\}$ defined by

$$\mathbf{q}_{k+1} = A\mathbf{q}_k \quad \text{for } k = 1, 2, 3, \ldots$$

and for a given $\mathbf{q}_0 \in \mathbb{R}^3$.

Show that $\lim_{k\to\infty} \mathbf{q}_k = \mathbf{0}$ for all $\mathbf{q}_0 \in \mathbb{R}^3$.

Problem 4.

(a) (8 points) Suppose λ is a complex eigenvalue (that is $\operatorname{Im}\lambda \neq 0$) of an $n \times n$ real matrix A, and $\mathbf{v}$ is an eigenvector corresponding to λ. Show that $\overline{\lambda}$ is also an eigenvalue of A, and $\overline{\mathbf{v}}$ is an eigenvector corresponding to $\overline{\lambda}$.

(b) (12 points) Find the eigenspace corresponding to each eigenvalue of

$$A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}.$$

Problem 5. (10 points) Consider a 3×3 matrix

$$A = \begin{bmatrix} q_{11}(s) & q_{12}(s) & q_{13}(s) \\ q_{21}(s) & q_{22}(s) & q_{23}(s) \\ q_{31}(s) & q_{32}(s) & q_{33}(s) \end{bmatrix}$$

that depends on a real parameter s, where $q_{jk} : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree three in s for $j, k = 1, 2, 3$. Now consider those s such that A is invertible, and for such s let

$$A^{-1} = \begin{bmatrix} r_{11}(s) & r_{12}(s) & r_{13}(s) \\ r_{21}(s) & r_{22}(s) & r_{23}(s) \\ r_{31}(s) & r_{32}(s) & r_{33}(s) \end{bmatrix}.$$

What kind of functions are $r_{jk} : \mathbb{R} \to \mathbb{R}$ for $j, k = 1, 2, 3$? For instance, are they polynomials, if yes what are their degree? Or, are they rational functions (that is can they be expressed as the ratio of two polynomials), if yes what are the degrees of the polynomials that appear in the numerators and the denominators of the rational functions? Support your answer.

Problem 6. For every $n \times n$ matrix A, there exist $n \times n$ invertible matrices V, W such that

$$WAV = D \tag{1}$$

is a diagonal matrix. By making use of this fact, in parts **(a)** and **(b)** below, either prove that the statement is true, or otherwise give a counter example proving that the statement is false.

(a) (5 points) A and D in (1) necessarily have the same eigenvalues.

(b) (5 points) Letting $T(\mathbf{x}) = A\mathbf{x}$, there exist two bases B and C for $\mathbb{R}^n$ such that

$$[T(\mathbf{x})]_C = D[\mathbf{x}]_B \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

where A and D are matrices as in (1) (in particular D is a diagonal matrix).