MATH 171B: Numerical Optimization

Instructor: Emre Mengi

Spring Quarter 2009
Midterm Exam #2
Wednesday May 20, 2009

NAME ______________________

#1	20	
#2	25	
#3	35	
#4	20	
Total	100	

- Duration: 55 minutes.
- No calculators.
- Show your work.
- Put your name in the box above.

Question 1. (20 points) Consider the quadratic polynomial $Q : \mathbf{R}^n \to \mathbf{R}$

$$Q(x) = x^T x.$$

Show that the iterate $x_1 = x_0 + p_0$ generated by the pure Newton's method (*i.e.* the step-length $\alpha_0 = 1$) is the unique local minimizer of Q for all initial points $x_0 \in \mathbf{R}^n$.

Question 2. Given a differentiable function $f : \mathbf{R}^n \to \mathbf{R}$, a point $\bar{x} \in \mathbf{R}^n$ and a search direction $\bar{p} \in \mathbf{R}^n$.

(a) (10 points) State the Armijo sufficient decrease condition for $f(x)$ at $\bar{x}$ in the direction of $\bar{p}$.

(b) (15 points) The graph of the function $\phi(\alpha) = f(\bar{x} + \alpha\bar{p})$ with $\phi'(0) = -0.8$ is provided overleaf. On this figure plot the graph of the line

$$\ell(\alpha) = f(\bar{x}) + \mu_1 \alpha \nabla f(\bar{x})^T \bar{p}$$

for $\mu_1 = 0.25$. Shade the intervals of α values satisfying the Armijo sufficient decrease condition on the horizontal axis.

Question 3. Consider the matrix

$$A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}.$$

(a) (15 points) Find the orthogonal eigenvalue decomposition of $A = V \Lambda V^T$ where Λ is a diagonal matrix and V is an orthogonal matrix, *i.e.* $V^T V = I$.

(b) (10 points) Let λ_1, λ_2 be the eigenvalues of A. Find a matrix $\tilde{A}$ with the same eigenvectors as A, but with eigenvalues $\tilde{\lambda}_i = |\lambda_i|$, $i = 1, 2$.

(c) (10 points) Consider a function $f : \mathbf{R}^2 \to \mathbf{R}$ and a point $\bar{x} \in \mathbf{R}^2$ such that $\nabla^2 f(\bar{x}) = A$ and $\nabla f(\bar{x}) = [-4 \;\; 4]^T$. Compute the modified Newton direction p_m for f at $\bar{x}$ by replacing the Hessian A with $\tilde{A}$. Is p_m a descent direction?

Question 4. (20 points) Given a twice continuously differentiable function $f : \mathbf{R}^n \to \mathbf{R}$. Suppose that the Hessian $\nabla^2 f(x)$ is positive definite at all $x \in \mathbf{R}^n$. Show that the sufficient decrease condition

$$\underbrace{(x_{k+1} - x_k)^T}_{s_k^T} \underbrace{(\nabla f(x_{k+1}) - \nabla f(x_k))}_{y_k} > 0$$

is satisfied for all pair of points $x_{k+1}, x_k \in \mathbf{R}^n$ such that $x_{k+1} \neq x_k$. (Hint : Apply Taylor's theorem with integral remainder to $\nabla f(x)$ in order to relate $\nabla f(x_{k+1})$ and $\nabla f(x_k)$. You may also need to refer to the following result,

given $g : \mathbf{R} \to \mathbf{R}$ such that $g(t) > 0$ for all $t \in [0, 1] \implies \int_0^1 g(t)dt > 0$)

Graph for question 2(b)