Math 409/509: Optimization

Midterm - Spring 2011

Duration : 75 minutes

Name ____________________

Student ID ____________________

Signature ____________________

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- Put your name and student ID in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.

Question 1. To compute $\frac{1}{\sqrt{5}}$ one approach is to apply Newton's method to function $f(x) = x^2 - \frac{1}{5}$.

- **(a)** (5 points.) Find the sequence $\{x_k\}$ resulting from the application of Newton's method to $f(x) = x^2 - \frac{1}{5}$.
- **(b)** (10 points.) Derive the order of convergence for the Newton sequence $\{x_k\}$ defined in part **(a)** assuming $\lim_{k\to\infty} x_k = \frac{1}{\sqrt{5}}$.
- **(c)** (15 points) Consider in general Newton's method for a zero of a univariate function $f : \mathbb{R} \to \mathbb{R}$ that is continuously differentiable. Let p_k be the Newton step associated with a point x_k at which both $f(x_k)$ and $f'(x_k)$ are nonzero. Show that
$$\frac{d}{d\alpha}|f(x_k + \alpha p_k)|\bigg|_{\alpha=0} < 0.$$

Question 2. Consider a multivariate function $f : \mathbb{R}^n \to \mathbb{R}$ that is twice continuously differentiable.

- **(a)** (5 points) State the second order sufficient conditions so that if a point $x_* \in \mathbb{R}^n$ satisfies these conditions, then it is a local minimizer of f.
- **(b)** (10 points) Specifically consider the function
$$f(x_1, x_2) = 3x_1^2 - x_1x_2 + x_2^2 + 7x_1 - 3x_2 + 4.$$
Find the stationary point of $f(x)$ and classify the stationary point as local a minimizer or not.
- **(c)** (15 points) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a quadratic function of the form
$$f(x) = \frac{1}{2}x^TAx + b^Tx + c$$
where A is symmetric and positive semidefinite. Prove that any stationary point of $f(x)$ is a global minimizer.

Question 3. (20 points) Let $x_k, x_{k+1} \in \mathbb{R}^n$ be two consecutive estimates generated by a quasi-Newton method for the unconstrained optimization of a twice-continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. Quasi-Newton methods typically form a symmetric matrix H_{k+1} that approximates $\left(\nabla^2 f(x_{k+1})\right)^{-1}$ and satisfies the secant equation
$$H_{k+1}y_k = s_k \tag{1}$$
where $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$ and $s_k = x_{k+1} - x_k$.

Suppose also that you are given a symmetric H_k which is an approximation for $\left(\nabla f^2(x_k)\right)^{-1}$ used at the previous iteration. Find a rank-one update formula of the form
$$H_{k+1} = H_k + \sigma uu^T$$
where H_{k+1} is symmetric and satisfies (1), $\sigma \in \mathbb{R}$, and $u \in \mathbb{R}^n$. In particular you need to determine σuu^T in terms s_k, y_k, and H_k.

Question 4. Consider the nonlinearly constrained problem

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^2} \quad & 3x_1^2 + (x_2-1)^2 \\ \text{subject} \quad & x_2 - \sin(x_1) = 0. \end{aligned} \tag{2}$$

(a) (10 points) Compute the tangent cone and the null space of the constraint Jacobian at $\bar{x} = (0{,}0)^T$ for problem (2). Is the constraint qualification satisfied at $\bar{x}$?

(b) (10 points) Write down the first order optimality conditions for problem (2) at point $\bar{x} = (0, 0)$. Is $(0, 0)$ a local minimizer?