# Math 304: Numerical Methods

Instructor: Emre Mengi

Spring Semester 2010
Midterm Exam #2
Wednesday May 5, 2010

Name ____________________

Student ID ____________________

Signature ____________________

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| #1 | 20 | |
| #2 | 20 | |
| #3 | 20 | |
| #4 | 20 | |
| #5 | 20 | |
| Total | 100 | |

- Put your name, student ID and signature in the boxes above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.

**Question 1.** In both parts $\|\cdot\|$ denotes the Euclidean length, *i.e.* $\|x\| = \sqrt{x^T x}$.

**(a)** (10 points) Find $x \in \mathbf{R}^2$ such that

$$\left\| \begin{bmatrix} 1 & -3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} x - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\|$$

is as small as possible by exploiting the QR factorization

$$\begin{bmatrix} 1 & -3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & 2\sqrt{3} \\ 0 & 0 \end{bmatrix}.$$

**(b)** (10 points) Let $A \in \mathbf{R}^{m \times n}$ with $m > n$ and $b \in \mathbf{R}^m$. Show that the $x$ value making $\|Ax - b\|$ as small as possible is unique if and only if $\dim(\text{Col}(A)) = n$.

**Question 2.** This questions concerns the integral

$$\int_0^2 x^2 f(x)\, dx \tag{1}$$

**(a)** (10 points) Find the Newton-cotes formula

$$w_0 f(x_0) + w_1 f(x_1)$$

for the integral (1) with two equally-distant nodes $x_0 = 0$ and $x_1 = 2$. Recall that to derive the formula you need to replace $f(x)$ with $f_n(x)$, which is the Lagrange polynomial passing through the points $(x_k, f(x_k))$ for $k = 0, 1$.

**(b)** (10 points) Determine $a$ and $b$ so that the degree of exactness of the quadrature formula of the form $af(b)$ (with only one node, *i.e.* $n = 0$) for the integral (1) is as large as possible.

**Question 3.** The QR algorithm is one of the standard approaches to compute the eigenvalues of a matrix $A \in \mathbf{R}^{n \times n}$. Below a pseudocode is provided for the QR algorithm. It generates a sequence of matrices $\{A_k\}$ that usually converges to an upper triangular matrix in the limit as $k \to \infty$.

```
A_0 ← A
for k = 0, 1, ... do
    Compute a QR factorization A_k = Q_k R_k
    A_{k+1} ← R_k Q_k
end for
```

**(a)** (10 points) Apply one iteration of the QR algorithm to the matrix $A$ provided below.

$$A = \begin{bmatrix} 3 & -2 \\ 4 & 6 \end{bmatrix}$$

**(b)** (10 points) It can be shown that under reasonable assumptions for a symmetric matrix $A$ as $k \to \infty$ the matrix

$$\hat{Q}_k = Q_0 Q_1 \dots Q_{k-1}$$

approach an orthogonal matrix $Q = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix}$ such that $q_1, q_2, \dots, q_n$ are the eigenvectors associated with the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A$.

Show that

$$Q^T A Q = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

where $\Lambda$ is a diagonal matrix.

**Question 4.** (20 points) Define the iterates of Newton's method for root-finding when applied to the function $f(x) = (x-2)(x-1)^2$. Derive the order of convergence for every Newton sequence that converges to a root of $f(x) = (x-2)(x-1)^2$ (i.e., derive the order of convergence for each zero of $f(x)$).

**Question 5.** This question concerns Newton's method for a root of a vector-valued function $F : \mathbf{R}^n \to \mathbf{R}^n$. Assume that $F$ is continuously differentiable and let $\bar{x}$ be a point at which the Jacobian matrix $F'(\bar{x})$ is nonsingular. Assume that $\|\cdot\|$ denotes the Euclidean length of a vector, *i.e.* $\|x\| = \sqrt{x^T x}$.

**(a)** (10 points) If $\psi : \mathbf{R} \to \mathbf{R}$ is the univariate function $\psi(\alpha) = \|F(\bar{x} + \alpha p)\|$, find an expression for $\psi'(\alpha)$ in terms of $p$, $F(\bar{x} + \alpha p)$ and $F'(\bar{x} + \alpha p)$.

**(b)** (10 points) Suppose that $p$ is the direction generated by Newton's method at $\bar{x}$ for finding a root of $F(x)$. Show that $\psi'(0) \leq 0$.
(Note : $\psi'(\alpha)$ is the directional derivative of $\|F(\bar{x} + \alpha p)\|$ along the direction $p$; $\psi'(0) < 0$ means that along the Newton direction $p$ the Euclidean length of $F(x)$ is decreasing at $\bar{x}$.)