# Math 107: Introduction to Linear Algebra

## Midterm 2, Part 2 - Spring 2020

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|---|---|---|
| #1 | 50 | |
| #2 | 50 | |
| $\sum$ | 100 | |

- The following time-frames are reserved for the questions.
  - Question 1, 19:00 - 19:35
  - Question 2, 19:40 - 20:15

- You must return your solution to each question by the end of the time-frame reserved for the question.

**Question 1.**

Let $A = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$ such that $\det A = 12$.

**(a)** **(9 points)** Determine the dimension of the column space of $A$. (That is determine the rank of $A$.) Justify your answer.

**(b)** **(8 points)** Calculate the determinant of $\begin{bmatrix} a-1 & 1 & 2 \\ b-2 & 3 & 4 \\ c-3 & 5 & 6 \end{bmatrix}$.

**(c)** **(8 points)** Calculate the determinant of $\begin{bmatrix} a & 3 & 3 \\ b & 7 & 6 \\ c & 11 & 9 \end{bmatrix}$.

**(c)** **(25 points)** Calculate the first row of $A^{-1}$.

Solution

**(a)**

Since $\det A \neq 0$, the matrix $A$ is invertible. This in turn implies that the columns of $A$ are linearly independent.

Hence, three columns of $A$ form a basis for Col $A$, that is

$$\text{Rank } A = \dim \text{Col } A = 3.$$

**(b)**

Note that $12 = \det A = \det A^T$. Note also, letting $B = \begin{bmatrix} a-1 & 1 & 2 \\ b-2 & 3 & 4 \\ c-3 & 5 & 6 \end{bmatrix}$, we have $\det B = \det B^T$.

Additionally,

$$A^T = \begin{bmatrix} a & b & c \\ 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \longrightarrow_{r_1 := r_1 - \frac{1}{2} r_3} \begin{bmatrix} a-1 & b-2 & c-3 \\ 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = B^T.$$

As the row replace operation does not change the determinant, we deduce

$$\det \underbrace{\begin{bmatrix} a-1 & 1 & 2 \\ b-2 & 3 & 4 \\ c-3 & 5 & 6 \end{bmatrix}}_{B} = \det B^T = \det A^T = 12.$$

**(c)**

Let $C = \begin{bmatrix} a & 3 & 3 \\ b & 7 & 6 \\ c & 11 & 8 \end{bmatrix}$, and observe that

$$A^T = \begin{bmatrix} a & b & c \\ 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \quad \longrightarrow_{r_2 := r_2 + r_3,\ r_3 := \frac{3}{2} r_3} \quad \begin{bmatrix} a & b & c \\ 3 & 7 & 11 \\ 3 & 6 & 9 \end{bmatrix} = C^T.$$

The row-replace operation above does not change the determinant, whereas, as a result of multiplying the third row by 3/2, the determinant is also multiplied by 3/2. Consequently,

$$\det \underbrace{\begin{bmatrix} a & 3 & 3 \\ b & 7 & 6 \\ c & 11 & 9 \end{bmatrix}}_{C} \; = \; \det C^T \; = \; \frac{3}{2} \det A^T \; = \; \frac{3}{2} \det A \; = \; 18.$$

**(d)**

Recall that by the Cramer's Rule

$$A^{-1} \quad = \quad \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix},$$

where $C_{ij}$ denotes the $(i, j)$-cofactor of $A$.

For the particular matrix $A$ at hand,

$$C_{11} = \det \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = (3)(6) - (4)(5) = -2$$

$$C_{21} = (-1)\det \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} = -\{(1)(6) - (2)(5)\} = 4$$

$$C_{31} = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (2)(3) = -2$$

. Consequently, the first row of $A^{-1}$ is given by

$$\frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & C_{31} \end{bmatrix} \quad = \quad \frac{1}{12} \begin{bmatrix} -2 & 4 & -2 \end{bmatrix} \quad = \quad \begin{bmatrix} -1/6 & 1/3 & -1/6 \end{bmatrix}$$

**Question 2.**

Let

$$A = \begin{bmatrix} 3 & -3 \\ 3 & 5 \\ 3 & 5 \\ 3 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}.$$

Moreover, let $\mathbf{a}_1$ and $\mathbf{a}_2$ be the first and the second columns of $A$, respectively.

**(a) (30 points)** Find the orthogonal projection of $\mathbf{x}$ onto $W = \text{span}\{\mathbf{a}_1, \mathbf{a}_2\}$.

**(b) (20 points)** Find a QR factorization of $A$.

SOLUTION

(a)

Observe that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is not orthogonal, as indeed

$$\mathbf{a}_1 \cdot \mathbf{a}_2 \;=\; (3)(-3) + (3)(5) + (3)(5) + (3)(-3) \;=\; 12 \;\neq\; 0. \tag{1}$$

To find the orthogonal projection, we first compute an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for $W = \text{span}\{\mathbf{a}_1, \mathbf{a}_2\}$ by applying Gram-Schmidt process. In particular,

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{a}_1 \\ \mathbf{u}_2 &= \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1, \end{aligned}$$

where

$$\begin{aligned} \mathbf{a}_2 \cdot \mathbf{u}_1 &= 12 \quad \text{(as calculated in (1) above)}, \\ \mathbf{u}_1 \cdot \mathbf{u}_1 &= (3)(3) + (3)(3) + (3)(3) + (3)(3) = 36. \end{aligned}$$

Consequently,

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ 5 \\ 5 \\ -3 \end{bmatrix} - \left(\frac{12}{36}\right) \cdot \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix}.$$

Finally, the orthogonal projection of $\mathbf{x}$ onto $W$ is given by

$$\text{proj}_W \mathbf{x} \;=\; \left(\frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right) \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} + \left(\frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right) \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix}.$$

Noting that

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_1 &= 36, \\ \mathbf{x} \cdot \mathbf{u}_1 &= (3)(1) + (3)(4) + (3)(0) + (3)(1) = 18, \\ \mathbf{u}_2 \cdot \mathbf{u}_2 &= (-4)(-4) + (4)(4) + (4)(4) + (-4)(-4) = 64, \\ \mathbf{x} \cdot \mathbf{u}_2 &= (1)(-4) + (4)(4) + (0)(4) + (1)(-4) = 8, \end{aligned}$$

we obtain

$$\text{proj}_W\,\mathbf{x} \;=\; \left(\frac{18}{36}\right)\begin{bmatrix}3\\3\\3\\3\end{bmatrix} + \left(\frac{8}{64}\right)\begin{bmatrix}-4\\4\\4\\-4\end{bmatrix} = \begin{bmatrix}1\\2\\2\\1\end{bmatrix}.$$

**(b)**
As discussed in the lecture notes, the $Q$ and $R$ factors of a QR factorization of $A$ are given by

$$Q = \begin{bmatrix} \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} & \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \end{bmatrix} \quad \text{and} \quad R = Q^T A.$$

Then, as $\|\mathbf{u}_1\| = 6$ and $\|\mathbf{u}_2\| = 8$, it follows from part **(a)** that

$$Q = \begin{bmatrix} 3/6 & -4/8 \\ 3/6 & 4/8 \\ 3/6 & 4/8 \\ 3/6 & -4/8 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

Moreover,

$$R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 3 & 5 \\ 3 & 5 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 0 & 8 \end{bmatrix}.$$

Hence,

$$\underbrace{\begin{bmatrix} 3 & -3 \\ 3 & 5 \\ 3 & 5 \\ 3 & -3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 6 & 2 \\ 0 & 8 \end{bmatrix}}_{R}$$

is a QR factorization of $A$.