MATH 304: Numerical Methods - I

Midterm II - Spring 2012
Duration : 90 minutes

NAME ____________________

STUDENT ID ____________________

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- Put your name and student ID in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book, but an open-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.

Question 1.

(a) (10 points) The hikers on a mountain measure the temperature at various hours of the day as listed in the table below.

time	$t_1 = 1$	$t_2 = 6$	$t_3 = 9$
temperature	$T_1 = 5$	$T_2 = 12$	$T_3 = 13$

Pose the problem of finding the line $\tau(t) = x_1 t + x_0$ (representing the temperature τ as a function of time t) minimizing

$$\sqrt{\sum_{i=1}^{3} (\tau(t_i) - T_i)^2}$$

with respect to the unknows $x_0, x_1 \in \mathbf{R}$ as a least-squares problem of the form $\boxed{\text{minimize}_x \|Ax - b\|_2}$.

(b) (10 points) Find $x \in \mathbb{R}^2$ such that

$$\left\| \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 8 \end{bmatrix} x - \begin{bmatrix} 7 \\ 10 \\ 15 \end{bmatrix} \right\|$$

is as small as possible by exploiting the QR factorization

$$\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 5\sqrt{3} \\ 0 & 3\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

Question 2. Write a pseudocode to calculate the inverse $A^{-1} \in \mathbb{R}^{n \times n}$ of a non-singular matrix $A \in \mathbb{R}^{n \times n}$ satisfying

$$A \cdot A^{-1} = I_n$$

where I_n denotes the $n \times n$ identity matrix. It is essential that the number of flops your pseudocode performs is proportional to n^3.

Question 3. Derive a quadrature formula for the integral

$$\int_0^4 f(x)\, dx$$

using $x_0 = 1$ and $x_1 = 3$ as the nodes with degree-of-exactness greater than or equal to one.

Question 4.
Find the monic orthogonal polynomial $q_2(x) = x^2 + \alpha_1 x + \alpha_0$ of degree two satisfying

$$\int_0^1 q_2(x)p(x)\,dx = 0$$

for all $p(x) \in \mathbb{P}_1$.

Question 5. Let $A, B \in \mathbb{R}^{n\times n}$. A scalar $\mu \in \mathbb{R}$ is called a *finite real eigenvalue* of $\mathcal{L}(\lambda) := A + \lambda B$ (where λ is a scalar) if there exists a nonzero $v \in \mathbb{R}^n$ such that

$$(A + \mu B)v = 0.$$

The vector v is called an eigenvector associated with μ.

(a) (10 points) Set up a system of nonlinear equations whose solution set is precisely the set of eigenpairs (μ, v) of $\mathcal{L}(\lambda)$ where μ is a finite real eigenvalue of $\mathcal{L}(\lambda)$ and v is a unit eigenvector associated with μ.

(b) (10 points) Derive the Newton sequence $\{x_k\}$ for the system of nonlinear equations in part **(a)**. In particular determine how x_{k+1} and x_k are related?