MATH 107: Introduction to Linear Algebra

Midterm 1 - Spring 2020
Duration : 100 minutes

#1	20	
#2	20	
#3	15	
#4	25	
#5	20	
Σ	100	

NAME ____________________

STUDENT ID ____________________

SIGNATURE ____________________

- Put your name, student ID and signature in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.
- Mark your section below.

  SECTION 1 (EMRE MENGI TUTH 13:00-14:15) ____________________
  SECTION 2 (EMRE MENGI, MW 11:30-12:45) ____________________
  SECTION 3 (EMRE MENGI, TUTH 8:30-9:45) ____________________
  SECTION 4 (MEHMET SARIDERELI, TUTH 14:30-15:45) ____________________
  SECTION 5 (AYBERK ZEYTIN, TUTH 8:30-9:45) ____________________

Solutions

Problem 1. (20 points) Let us consider a system of linear equations

$$Ax = b, \tag{1}$$

where $b \in \mathbb{R}^3$ and the coefficient matrix A is 3×5. Moreover, A can be reduced to the echelon form

$$\begin{bmatrix} 1 & 2 & -1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

by applying row operations, while the vector $p = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \end{bmatrix}^T$ is a solution of the system of linear equations in (1), that is it satisfies $Ap = b$.

Write down the general solution of the system of linear equations in (1).

Find solution of $Ax = 0$

pivot positions

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow[r_1 := r_1 + r_2]{r_2 := r_2 - r_3} \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

$$\begin{cases} x_2, x_5 \text{ are free} \\ x_4 = -3x_5, \quad x_3 = -x_5, \quad x_1 = -2x_2 + 2x_5 \end{cases}$$

Solution set for $Ax = 0$

$$S = \left\{ \begin{bmatrix} -2x_2 + 2x_5 \\ x_2 \\ -x_5 \\ -3x_5 \\ x_5 \end{bmatrix} \middle| x_5 \in \mathbb{R} \right\} = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \\ 1 \end{bmatrix} \middle| x_2, x_5 \in \mathbb{R} \right\}$$

Solution set for $Ax = b$

$$\{ p + v \mid v \in S \}$$

$$= \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \\ +1 \end{bmatrix} \middle| x_2, x_5 \in \mathbb{R} \right\}$$

Problem 2. (20 points) Find the values of $a, b \in \mathbb{R}$ so that

$$\begin{bmatrix} 3 \\ -1 \\ 2 \\ 2 \end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ b \\ 4 \end{bmatrix} \right\}.$$

Find $a, b \in \mathbb{R}$ such that

$$(+) \quad c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ a \\ b \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 2 \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & a \\ 1 & -2 & b \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 2 \end{bmatrix}$$

has a solution c_1, c_2, c_3.

Now reduce the augmented matrix (by applying row operations)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & a & -1 \\ 1 & -2 & b & 2 \\ 1 & 2 & 4 & 2 \end{array}\right] \xrightarrow[\substack{r_2 := r_2 - 2r_1 \\ r_3 := r_3 - r_1 \\ r_4 := r_4 - r_1}]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & a-2 & -7 \\ 0 & -3 & b-1 & -1 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

$$\xrightarrow[\substack{r_3 := r_3 - 3r_2 \\ r_4 := r_4 + r_2}]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & a-2 & -7 \\ 0 & 0 & b-3a+5 & 20 \\ 0 & 0 & a+1 & -8 \end{array}\right]$$

$$\begin{pmatrix}\text{assuming} \\ b-3a+5 \neq 0\end{pmatrix} \xrightarrow[r_4 := r_4 - \frac{(a+1)}{b-3a+5} r_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & a-2 & -7 \\ 0 & 0 & b-3a+5 & 20 \\ 0 & 0 & 0 & -8 - \frac{a+1}{b-3a+5} 20 \end{array}\right]$$

It follows that (+) has a solution if and only if

$$b - 3a + 5 \neq 0 \quad \text{and} \quad \underbrace{8 + \frac{a+1}{b-3a+5} 20 = 0}$$

(plug in $a = 2b+15$ to obtain ② $-5b - 40 \neq 0$)

$\Leftrightarrow -2(b-3a+5) = 5(a+1)$

$\Leftrightarrow a = 2b + 15$ ①

that is (+) has a solution if and only if

$$a = 2b+15 \quad \text{and} \quad b \neq -8$$

Problem 3. Let T be the linear transformation defined by

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \qquad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 + 2x_3 \\ x_3 \end{bmatrix}.$$

When answering parts **(a)** and **(b)** below, explain your reasoning.

(a) (8 points) Is T one-to-one?

Standard matrix of T

$$(*) \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Check whether $Ax=0$ has a nontrivial solution.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\substack{r_1 := r_1 - 3r_3 \\ r_1 := r_1 - 2r_2}]{r_2 := r_2 - 2r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

No, the only solution is $x=0$, so the columns of A are linearly independent. This implies T is one-to-one.

(b) (7 points) Is T onto?

Since every row of A in $(*)$ has a pivot position,

$\underbrace{Ax}_{T(x)}=b$ has a solution for all b.

This implies T is onto.

Problem 4.

(a) (5 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a one-to-one, onto linear transformation with the standard matrix M, and $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ be the inverse of T with the standard matrix N. How are the matrices M and N related?

$$MN = I_n \quad \text{and} \quad NM = I_n$$

(b) (20 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\3\end{bmatrix}.$$

Find the standard matrix of $T^{-1}(\mathbf{x})$.

$T(x) = Ax$, where

$$A = \left[T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \right] = \begin{bmatrix}1 & 2 & 1\\-1 & 0 & 1\\1 & 1 & 3\end{bmatrix}$$

Then $T^{-1}(x) = A^{-1}x$, that is A^{-1} is the standard matrix of T^{-1}!

Computation of A^{-1}

$$\left[\underbrace{\begin{matrix}1 & 2 & 1\\-1 & 0 & 1\\1 & 1 & 3\end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix}}_{I_3}\right] \xrightarrow[r_3 := r_3 - r_1]{r_2 := r_2 + r_1} \left[\begin{matrix}1 & 2 & 1\\0 & 2 & 2\\0 & -1 & 2\end{matrix} \;\middle|\; \begin{matrix}1 & 0 & 0\\1 & 1 & 0\\-1 & 0 & 1\end{matrix}\right]$$

$$\xrightarrow{r_3 := r_3 + (\frac{1}{2})r_2} \left[\begin{matrix}1 & 2 & 1\\0 & 2 & 2\\0 & 0 & 3\end{matrix} \;\middle|\; \begin{matrix}1 & 0 & 0\\1 & 1 & 0\\-1/2 & 1/2 & 1\end{matrix}\right]$$

$$\xrightarrow{r_3 := (1/3) r_3} \left[\begin{matrix}1 & 2 & 1\\0 & 2 & 2\\0 & 0 & 1\end{matrix} \;\middle|\; \begin{matrix}1 & 0 & 0\\1 & 1 & 0\\-1/6 & 1/6 & 1/3\end{matrix}\right]$$

$$\xrightarrow[r_1 := r_1 - r_3]{r_2 := r_2 - 2r_3} \left[\begin{matrix}1 & 2 & 0\\0 & 2 & 0\\0 & 0 & 1\end{matrix} \;\middle|\; \begin{matrix}7/6 & -1/6 & -1/3\\4/3 & 2/3 & -2/3\\-1/6 & 1/6 & 1/3\end{matrix}\right]$$

$$\xrightarrow[r_1 := r_1 - 2r_2]{r_2 := (1/2) r_2} \left[\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix} \;\middle|\; \begin{matrix}-1/6 & -5/6 & 1/3\\2/3 & 1/3 & -1/3\\-1/6 & 1/6 & 1/3\end{matrix}\right]$$

$$A^{-1} = \begin{bmatrix}-1/6 & -5/6 & 1/3\\2/3 & 1/3 & -1/3\\-1/6 & 1/6 & 1/3\end{bmatrix}$$

is the standard matrix of T^{-1}.

Problem 5. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_k \in \mathbb{R}^n$ be given vectors.

(a) (15 points) Show that if $\{T(\mathbf{a}_1), T(\mathbf{a}_2), \ldots, T(\mathbf{a}_k)\}$ is linearly independent, then $\{\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_k\}$ is also linearly independent.

Suppose

$$c_1 a_1 + \cdots + c_k a_k = 0$$

for some $c_1, \ldots, c_k \in \mathbb{R}$.

But then by linearity of T

$$T(c_1 a_1 + \cdots + c_k a_k) = T(0) = 0$$

and

$$c_1 T(a_1) + \cdots + c_k T(a_k) = 0.$$

Since $\{T(a_1), \ldots, T(a_k)\}$ is linearly independent, we have $c_1 = c_2 = \cdots = c_k = 0$, proving that $\{a_1, \ldots, a_k\}$ is linearly independent

(b) (5 points) Prove or disprove the following statement:

- If $\{\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_k\}$ is linearly independent, then $\{T(\mathbf{a}_1), T(\mathbf{a}_2), \ldots, T(\mathbf{a}_k)\}$ is also linearly independent.

(Note: To prove the statement, you must show its validity for every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and every $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_k \in \mathbb{R}^n$. To disprove, it is sufficient to give one counter example.)

Consider

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

The set

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

is clearly independent, but

$$\left\{ T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right), T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

is linearly dependent.

This shows that the statement is false