SOLUTIONS TO MIDTERM I

Q1

(a)

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

$$= \max_{\left\|\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\|_2=1} \left\| \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_2$$

$$= \max_{\left\|\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\|_2=1} \left\| \begin{bmatrix} 2x_1 - 3x_2 \\ 3x_1 + 2x_2 \end{bmatrix} \right\|_2$$

$$= \max_{\left\|\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\|_2=1} \sqrt{13\,(x_1^2 + x_2^2)} = \sqrt{13}$$

(b)

$$\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v_1} = \sqrt{13} \left(\underbrace{\frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{u_1} \right)$$

Let

$$U = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$U^* \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} V = \begin{bmatrix} \sqrt{13} & 0 \\ 0 & \sqrt{13} \end{bmatrix}$$

Q2

(a)

Orthogonal projector onto S_1

$$P_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{bmatrix}$$

(b) Orthogonal projector onto S_2

$$I - P_1 = \cancel{\begin{bmatrix} 1/6 & 2/3 & 7/6 \\ 2/ & & \\ & & \end{bmatrix}}$$

$$\begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

Q3

$$\min \left\{ \|\Delta A\|_2 \mid \Delta A \in \mathbb{C}^{n\times n} \text{ s.t. } (A+\Delta A)x = \lambda x \ \exists x \neq 0 \right\}$$

$$= \min \left\{ \|\Delta A\|_2 \mid \Delta A \in \mathbb{C}^{n\times n} \text{ s.t. } \cancel{\operatorname{rank}(A+\Delta A) < n} \ \operatorname{rank}(A-\lambda I+\Delta A) < n \right\}$$

$$= \sigma_n (A - \lambda I) \quad \left(\begin{matrix}\text{OPTIMAL} \\ \text{RANK} \\ \text{THM}\end{matrix}\right)$$

Optimal ΔA_* is given by

$$\Delta A_* = -\sigma_n u_n v_n^*$$

where

$\sigma_n := \sigma_n(A - \lambda I)$

u_n, v_n are left and right singular vectors associated with σ_n

Q4

A, B are unitarily equivalent

① $\Longrightarrow$

A, B have the same set of singular values

$\exists\, U \in \mathbb{C}^{m\times m},\ V \in \mathbb{C}^{n\times n}$ unitary s.t.

$$A = U B V$$

Let B have the SVD

$$B = \tilde{U} \Sigma \tilde{V}^*$$

Then

$$A = \underbrace{U\tilde{U}}_{\tilde{\tilde{U}}} \Sigma \underbrace{\tilde{V}^* V}_{\tilde{\tilde{V}}^*}$$

is an SVD for A. Consequently A and B have the same singular values, i.e, diagonal entries of Σ.

(2) A, B have the same set of singular values

$$\Longrightarrow$$

A, B are unitarily equivalent

Let

$$A = U_1 \Sigma V_1^*$$
$$B = U_2 \Sigma V_2^*$$

be full SVDs. Then

$$\Sigma = U_2^* B V_2$$
$$\Longrightarrow$$
$$A = U_1 \Sigma V_1^* = U_1 U_2^* B V_2 V_1^* = \tilde{U} B \tilde{V}^*$$

where $\tilde{U} = U_1 U_2^*$ and $\tilde{V} = V_1 V_2^*$ are unitary. □

Q5

$R = 0$
for $j = 1, \dots, n$
  $q_{2j-1} = a_{2j-1}$, $q_{2j} = a_{2j}$
  for $i = 1, \dots, j-1$
    $r_{2i-1, 2j-1} = a_{2j-1}^* q_{2i-1}$ } $4n-1$ FLOPS
    $r_{2i, 2j} = a_{2j}^* q_{2i}$ } $4n-1$ FLOPS
    $r_{2i, 2j-1} = 0$
    $r_{2i-1, 2j} = 0$

$q_{2j-1} = q_{2j-1} - r_{2i-1,2j-1}\, q_{2i-1}$ } 4n FLOPS

$q_{2j} = q_{2j} - r_{2i,2j}\, q_{2i}$ } 4n FLOPS

end for

$r_{2j-1,2j-1} = \| q_{2j-1} \|_2$ } 4n FLOPS

$r_{2j,2j} = \| q_{2j} \|_2$ } 4n FLOPS

$q_{2j-1} = q_{2j-1} / r_{2j-1,2j-1}$ } 2n FLOPS

$q_{2j} = q_{2j} / r_{2j,2j}$ } 2n FLOPS

end for

TOTAL # FLOPS

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{j-1} 16n - 1 \right) + 10n$$

$$= \sum_{j=1}^{n} (16n-1)(j-1) + 10n$$

$$= \sum_{j=1}^{n} 16nj + O(n) = 16 \frac{n\, n(n+1)}{2} + O(n^2)$$

$$= 8n^3 + O(n^2)$$