# MATH 504: Numerical Methods - I

Midterm - Fall 2011
Duration : 90 minutes

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|---|---|---|
| #1 | 20 | |
| #2 | 20 | |
| #3 | 20 | |
| #4 | 20 | |
| #5 | 20 | |
| $\Sigma$ | 100 | |

NAME ____________________

STUDENT ID ____________________

SIGNATURE ____________________

- Put your name and student ID in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book, but an open-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.

**Question 1.** Consider the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}.$$

**(a)** Find the 2-norm of $A$.

**(b)** Calculate a singular value decomposition for $A$.

**Question 2.** Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix},$$

$\mathcal{S}_1 = \text{Range}(A)$, and $\mathcal{S}_2 = \mathcal{S}_1^\perp$. (Here $\mathcal{S}_1^\perp$ denotes the orthogonal complement of $\mathcal{S}_1$, that is $\mathcal{S}_1 \perp \mathcal{S}_1^\perp$ and $\mathcal{S}_1 \oplus \mathcal{S}_1^\perp = \mathbb{R}^3$.)

**(a)** Find the orthogonal projector onto $\mathcal{S}_1$.

**(b)** Find the orthogonal projector onto $\mathcal{S}_2$.

**Question 3.** A scalar $\lambda \in \mathbb{C}$ is called an eigenvalue of $A \in \mathbb{C}^{n\times n}$ if there exists a nonzero vector $v \in \mathbb{C}^n$ such that

$$Av = \lambda v.$$

Derive an expression for the matrix closest to $A$ with respect to the 2-norm with $\lambda$ as an eigenvalue. Formally determine $A + \Delta A_*$ where $\Delta A_*$ is a global minimizer for the problem

$$\text{minimize}\{\|\Delta A\|_2 \;\mid\; \Delta A \in \mathbb{C}^{n\times n} \text{ such that } (A + \Delta A)x = \lambda x \; \exists x \neq 0\}$$

Your expression must be in terms of singular values and singular vectors.

**Question 4.** Two matrices $A, B \in \mathbb{C}^{m\times n}$ are said to be unitarily equivalent if there exist unitary transformations $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that $A = UBV$.

Show that $A$ and $B$ are unitarily equivalent if and only if they have precisely the same set of singular values.

**Question 5.** Every matrix $A \in \mathbb{C}^{2n\times 2n}$ has a factorization of the form

$$A = \tilde{Q}R \tag{1}$$

where $R \in \mathbb{C}^{2n\times 2n}$ is upper triangular, and $\tilde{Q} \in \mathbb{C}^{2n\times 2n}$ is such that

**(i)** $\{\tilde{q}_{2j-1} \mid j = 1, \ldots, n\}$ is orthonormal,

**(ii)** $\{\tilde{q}_{2j} \mid j = 1, \ldots, n\}$ is orthonormal,

**(iii)** $\text{span}\{\tilde{q}_{2j-1} \mid j = 1, \ldots, n\} = \text{span}\{a_{2j-1} \mid j = 1, \ldots, n\}$, and

**(iv)** $\text{span}\{\tilde{q}_{2j} \mid j = 1, \ldots, n\} = \text{span}\{a_{2j} \mid j = 1, \ldots, n\}$

where $\tilde{q}_j, a_j \in \mathbb{C}^{2n}$ denote the $j$th columns of $\tilde{Q}$ and $A$, respectively.

Write down a pseudocode to compute a factorization of the form (1) for $A \in \mathbb{C}^{2n \times 2n}$. Also perform a flop count for your pseudocode.