Math 107: Introduction to Linear Algebra

Midterm 1 - Spring 2020

Duration : 100 minutes

#1	20	
#2	20	
#3	15	
#4	25	
#5	20	
$\sum$	100	

Name ____________________

Student ID ____________________

Signature ____________________

- Put your name, student ID and signature in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.
- Mark your section below.

  Section 1 (Emre Mengi TuTh 13:00-14:15) ____________________
  Section 2 (Emre Mengi, MW 11:30-12:45) ____________________
  Section 3 (Emre Mengi, TuTh 8:30-9:45) ____________________
  Section 4 (Mehmet Sariderelli, TuTh 14:30-15:45) ____________________
  Section 5 (Ayberk Zeytin, TuTh 8:30-9:45) ____________________

Problem 1. (20 points) Let us consider a system of linear equations

$$A\mathbf{x} = \mathbf{b}, \tag{1}$$

where $\mathbf{b} \in \mathbb{R}^3$ and the coefficient matrix A is 3×5. Moreover, A can be reduced to the echelon form

$$\begin{bmatrix} 1 & 2 & -1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

by applying row operations, while the vector $\mathbf{p} = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \end{bmatrix}^T$ is a solution of the system of linear equations in (1), that is it satisfies $A\mathbf{p} = \mathbf{b}$.

Write down the general solution of the system of linear equations in (1).

Problem 2. (20 points) Find the values of $a, b \in \mathbb{R}$ so that

$$\begin{bmatrix} 3 \\ -1 \\ 2 \\ 2 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ b \\ 4 \end{bmatrix} \right\}.$$

Problem 3. Let T be the linear transformation defined by

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \qquad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 + 2x_3 \\ x_3 \end{bmatrix}.$$

When answering parts **(a)** and **(b)** below, explain your reasoning.

(a) (8 points) Is T one-to-one?

(b) (7 points) Is T onto?

Problem 4.

(a) (5 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a one-to-one, onto linear transformation with the standard matrix M, and $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ be the inverse of T with the standard matrix N. How are the matrices M and N related?

(b) (20 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

Find the standard matrix of $T^{-1}(\mathbf{x})$.

Problem 5. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and $\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_k \in \mathbb{R}^n$ be given vectors.

(a) (15 points) Show that if $\{T(\mathbf{a}_1), T(\mathbf{a}_2), \dots, T(\mathbf{a}_k)\}$ is linearly independent, then $\{\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_k\}$ is also linearly independent.

(b) (5 points) Prove or disprove the following statement:

- If $\{\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_k\}$ is linearly independent, then $\{T(\mathbf{a}_1), T(\mathbf{a}_2), \dots, T(\mathbf{a}_k)\}$ is also linearly independent.

(Note: To prove the statement, you must show its validity for every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and every $\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_k \in \mathbb{R}^n$. To disprove, it is sufficient to give one counter example.)