MATH 304: Numerical Methods

Instructor: Emre Mengi

Spring Semester 2010
Midterm Exam #1
Wednesday March 24, 2010

#1	20	
#2	20	
#3	20	
#4	15	
#5	25	
Total	100	

NAME ____________________

STUDENT ID ____________________

SIGNATURE ____________________

- Put your name, student ID and signature in the boxes above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.

Question 1. (20 points) Given an $n \times n$ matrix A and $b \in \mathbf{R}^n$. Consider the following two Matlab routines both of which solve the linear system $A^n x = b$ for $x \in \mathbf{R}^n$.

```
function x = Solver1(A,b)

n = length(b);

[L,U] = lu(A);
for j = 1:n

  % forward substitution
  xhat = L\b;

  % back substitution
  x = U\xhat;

  b = x;
end

x = b;
return;
```

```
function x = Solver2(A,b)

n = length(b);

P = A;
for j = 2:n
  P = A*P;
end

[L,U] = lu(P);

% forward substitution
xhat = L\b;

% back substitution
x = U\xhat;

return;
```

Matlab uses forward substitution to solve a lower triangular system and back substitution to solve an upper triangular system. Calculate the *total number of flops* required by the routines `Solver1` and `Solver2` in terms of n. (You can use the big-O notation in your answers.)

Question 2. (20 points) In a population consisting of rabbits and foxes let r_n and f_n denote the number of rabbits and foxes at the beginning of the year n, respectively. The number of rabbits and foxes at the beginning of the next year are given by the formulas

$$r_{n+1} = 1.6 r_n - 0.3 f_n \quad \text{and} \quad f_{n+1} = 1.8 r_n + 0.1 f_n.$$

These two equations can be combined to yield the recurrence equation

$$\underbrace{\begin{bmatrix} r_{n+1} \\ f_{n+1} \end{bmatrix}}_{P_{n+1}} = \underbrace{\begin{bmatrix} 1.6 & -0.3 \\ 1.8 & 0.1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} r_n \\ f_n \end{bmatrix}}_{P_n}.$$

(a) (10 points) Find the *eigenvalues* and the associated *eigenspaces* of the matrix A.

(b) (5 points) Suppose that you apply the power iteration to the matrix A. Which eigenvalue and eigenvector of A would the power iteration converge (if it converges at all)?

(c) (5 points) Suppose that initially the number of rabbits and foxes are given as $r_0 = 10^{50}$ and $f_0 = 3 \times 10^{50}$. Find r_{50} and f_{50}.

Question 3. (20 points) Given an $n \times n$ matrix A and another $n \times n$ matrix $B = TAT^{-1}$ that is *similar* to A where T is an $n \times n$ invertible matrix. Indicate whether each of the following statements is true or false with a brief explanation.

(a) (7 points) The *characteristic polynomials* of A and B are the same.

(b) (7 points) The *eigenvalues* of A and B are the same.

(c) (6 points) The *eigenvectors* of A and B are the same.

Question 4. (15 points) An LU factorization for a 3×3 matrix A is given as

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}.$$

(a) (5 points) Calculate the *determinants* of L, U and A.

(b) (10 points) Solve the linear system

$$Ax = \begin{bmatrix} 3 \\ 4 \\ -3 \end{bmatrix}$$

for $x \in \mathbf{R}^3$ using *forward and back substitutions.*

Question 5. (25 points) Consider the vector v and the matrix A given below.

$$v = \begin{bmatrix} 4 \\ -1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 4 & 11 \\ -1 & 7 \\ 2 & 12 \\ 2 & 12 \end{bmatrix}$$

(a) (8 points) Find the *orthogonal projector* $P_v \in \mathbf{R}^{4\times 4}$ onto $\operatorname{span}\{v\}$. Is P_v invertible or not? Justify your answer.

(b) (9 points) Find an *orthogonal matrix* Q such that

$$Q \begin{bmatrix} 4 \\ -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c) (8 points) Compute the *QR factorization* of the matrix A of the form $A = QR$ where $Q \in \mathbf{R}^{4\times 4}$ is orthogonal and $R \in \mathbf{R}^{4\times 2}$ is upper triangular (*i.e.* $r_{ij} = 0$ for all i, j such that $i > j$).