MATH 107: Introduction to Linear Algebra

Midterm 1 - Spring 2017

Duration : 100 minutes

#1	15	
#2	20	
#3	15	
#4	25	
#5	25	
$\sum$	100	

NAME ________________

STUDENT ID ________________

SIGNATURE ________________

- Put your name, student ID and signature in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.
- Mark your section below.

SECTION 1 (EMRE MENGI TUTH 11:30-12:45) ________________

SECTION 2 (EMRE MENGI, TUTH 8:30-9:45) ________________

SECTION 3 (EMRE MENGI, MW 13:00-14:15) ________________

SECTION 4 (DOĞAN BILGE, MW 14:30-15:45) ________________

Problem 1.

(a) (10 points) Write down the solution set of the following linear system as the span of a set of vectors.

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 3 & 8 & 2 \\ 1 & 2 & 5 & 3 \end{bmatrix}}_{A} \mathbf{x} \;=\; \mathbf{0} \tag{1}$$

(b) (5 points) Let A be the 3×4 matrix as in (1) above and $\mathbf{b} \in \mathbb{R}^3$ be a vector such that $\mathbf{p} \;=\; \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}^T$ is a solution of $A\mathbf{x} = \mathbf{b}$. Find the solution set of $A\mathbf{x} = \mathbf{b}$.

Problem 2.

(a) (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ -3 & 1 & 5 & 2 \\ -5 & -2 & 1 & 7 \end{bmatrix}.$$

Find a subset of the columns of A that is as large as possible and that is linearly independent.

(b) (10 points) Find a vector $\mathbf{b} \in \mathbb{R}^3$ that does not belong to

$$\operatorname{span}\left\{ \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 12 \end{bmatrix} \right\}.$$

Problem 3.

(a) (5 points) Write down a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ that is one-to-one.

(b) (5 points) Write down a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\text{Range of } T \quad = \quad \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

(c) (5 points) Write down a transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ that is not linear and that is onto.

Problem 4.

(a) (15 points) Find the standard matrix of the transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^3, \quad T(\mathbf{x}) \ = \ T_2(T_1(\mathbf{x})),$$

where

- $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation such that $T_1(\mathbf{x})$ is the reflection of the vector $\mathbf{x} \in \mathbb{R}^2$ about the line $x_2 = \sqrt{3}x_1$,
- $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$ is the linear transformation such that

$$T_2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad T_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}.$$

(b) (10 points) Find the inverse of the following transformation.

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \qquad T(\mathbf{x}) \;=\; \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 + x_3 \end{bmatrix}$$

Problem 5.

(a) (9 points) Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one and onto linear transformation. By explicitly employing the definition of a linear transformation, show that $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is also linear.

(b) (8 points) Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Show that if $k > n$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependent.

(c) (8 points) Let A and B be $m \times n$ matrices. Show that if $(A - B)\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.