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Math 304 (Spring 2010)

Study Guide for Weeks 11-12
Homework 8 concerns one-step methods for the numerical solution of ordinary differential equations. Specifically in class we focused on the following topics. (In addition to Fausett's book another resource is the first chapter of Arieh Iserles' book "A first course in the numerical analysis of differential equations", which is available at the reserve desk in the library. However, this book is intended only for ***mathematically mature students***. Read this book only if you are very eager to gain a deep understanding of the numerical differentiation methods.)

- Euler's Method (Fausett - 12.1)
- Trapezoidal method
- Order of a numerical method for the solution of ordinary differential equations.
  - Order of Euler's method is one.
  - Order of trapezoidal method is two.
- Global convergence and error of a numerical method for the solution of ordinary differential equations.
  - Global error of Euler's method is proportional to the step-size h.
  - Global error of Trapezoidal method is proportional to the square of the step-size h^2.
- Adams-Bashforth methods, especially 2-step Adams-Bashforth method (Fausett - 12.3.1)

Homework 8 (due on May 26th, Wednesday by 4pm)
For questions **3** and **4** you will need to use the implementations of Euler's method (`Euler.m`) and 2-step Adams-Bashforth (`Adam_Bash.m`) methods provided on the course webpage for the numerical solution of the ODE

$$y'(t) = f(t, y(t)) \ \ t > a \ \text{ and } \ y(a) = y_0$$

on the interval $[a, b]$. Both of the m-files take five input parameters;

- fun - the name of the m-file that evaluates $f(t, y)$ at a given t and y
- a, b - the left and right end-points of the interval on which the solution of the ODE is sought
- h - the step size
- y0 - the initial condition.

They return two output parameters;

- yvec - the approximate solution vector at the discrete points
- tvec - vector of t values at which the approximate y values are computed.

For instance to solve the ODE

$$y' = 3t^2 y \;\; t > 0 \;\text{ and }\; y(0) = 1$$

on the unit interval $[0,1]$ using Euler's method with the step-size $h = 0.01$ type

```
[yvec, tvec] = Euler('fun_ODE',0,1,0.01,1);
```

Make sure to download the m-file `fun_ODE.m` that evaluates $f(t,y) = 3t^2 y$ before typing the Matlab command above. Also open and go through the m-file `fun_ODE.m` to understand its input and output parameters. You can plot the approximate solution by typing

```
plot(tvec,yvec,'b-*');
```

For questions **3** and **4** attach your Matlab outputs/figures as well as the print-outs of the m-files that you implemented.

1. Solve each of the following ordinary differential equations using

(i) Euler's method with $h = 0.5$ and **(ii)** 2-step Adams-Bashforth method with $h = 0.5$.

Additionally for each of the ODEs write down the nonlinear system resulting from an application of the trapezoidal method with $h = 0.5$. You must perform all calculations by hand.

$$\begin{array}{ll} \textbf{(ODE 1)} & y' = y^2 \;\; t > 0, \;\; y(0) = -1 \;\text{ on }\; [0,1] \\ \textbf{(ODE 2)} & y' = -2y\sin(2t) \;\; t > 0, \;\; y(0) = \mathrm{e} \;\text{ on }\; [0,1] \end{array}$$

2. Show that

(i) the order of backward Euler's method with the update rule

$$y_{k+1} = y_k + hf(t_{k+1}, y_{k+1})$$

is one,

(ii) the order of the trapezoidal method with the update rule

$$y_{k+1} = y_k + \frac{h}{2}\left(f(t_k, y_k) + f(t_{k+1}, y_{k+1})\right)$$

is two.

3. (Fausett A12.3, page 494) The velocity v of an object subject to the gravitational force and air resistance (proportional to v) satisfies the ODE

$$\frac{dv}{dt} = g - \frac{kv}{m}$$

where $g = 9.81$ m/s^2 is the gravitational acceleration, k is a constant for the air-resistance and m is the mass of the object.

(a) Suppose that the initial velocity $v(0) = 100$ m/s and $k/m = 0.05$ 1/s. Find the approximate solution on the time interval $[0,3]$ using Euler's method (in particular use the m-file `Euler.m` on the course webpage) with the step size $h = 0.001$. Plot the approximate solution.

(b) Repeat part **(a)** but with the initial velocity $v(0) = 0$ m/s and $k/m = 1.5$ 1/s.

4. The purpose of this question is to compare the accuracy of Euler's method and 2-step Adams-Bashforth method. Consider the ODE

$$y' = y\cos t \;\; t > 0, \;\; \text{and} \;\; y(0) = 1$$

with the exact solution $y(t) = e^{\sin t}$.

(a) Solve this ODE using Euler's method with $h = 0.1, 0.01, 0.001, 0.0001$ on the interval $[0, 2]$.

(i) Download the routines `generate_fun.m` and `fun_real_trig.m` from the course webpage. Plot the actual solution $y(t) = e^{\sin t}$ on the interval $[0, 2]$ by typing

```
>> [yreal, treal] = generate_fun('fun_real_trig',0,2,0.001);
>> plot(yreal, treal, 'r-*');
```

(ii) Plot the computed solutions for each of the h values above on the same figure.

(iii) Calculate the error at time $t = 2$ of the computed solution for each of the h values. Does the global error decay as expected in theory?

(b) Repeat part **(a)** but with the 2-step Adams-Bashforth instead of Euler's method.