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Math 304 (Spring 2010)

Study Guide for Weeks 9-10
Homework 7 concerns the topics listed below.

- (Discrete) least squares - problem definition (Fausett - 9.1)
- Solution of the least squares problem by QR factorization
- Interpolation, Lagrange and Newton forms (Fausett - 8.1)
- Numerical integration, Newton-Cotes; recall that Trapezoidal and Simpson's rule are special cases on which we put emphasis in class (Fausett - 11.2)
- Orthogonal Polynomials, in particular Legendre and Chebyshev polynomials (Fausett - 9.2.2-4)
- Gaussian Quadrature (Fausett - 11.3.1)

Homework 7 (due on May 7th, Friday by 4pm)
In Matlab question (question **3**) attach the Matlab outputs and print-outs of the Matlab routines that you implemented.

1. Given the following five points in $\mathbb{R}^2$.

$$p_1 = (\underbrace{-4}_{t_1}, \underbrace{-1}_{y_1}),\ p_2 = (\underbrace{-2}_{t_2}, \underbrace{-5}_{y_2}),\ p_3 = (\underbrace{-1}_{t_3}, \underbrace{-0.5}_{y_3})\ p_4 = (\underbrace{1}_{t_4}, \underbrace{2}_{y_4}) \text{ and } p_5 = (\underbrace{3}_{t_5}, \underbrace{1}_{y_5})$$

(a) Pose the problem of finding the line $\ell(t) = x_1 t + x_0$ minimizing

$$\sqrt{\sum_{i=1}^{5} (\ell(t_i) - y_i)^2}$$

with respect to the unknows x_0, x_1 as a least-squares problem in the form $\boxed{\text{minimize}_x \|Ax - b\|_2}$.

(b) Repeat part **(a)** for the polynomial of the form $P(t) = x_4 t^4 + x_3 t^3 + x_2 t^2 + x_1 t + x_0$ minimizing

$$\sqrt{\sum_{i=1}^{5} (P(t_i) - y_i)^2}$$

with respect to the unknowns x_0, x_1, x_2, x_3, x_4.

2. Given the least squares problem

$$\text{minimize}_{x \in \mathbb{R}^2} \left\| \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ -1 & -1 \\ 1 & 2 \end{bmatrix}}_{A} x - \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{b} \right\|_2$$

where A has the QR factorization

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ -1 & -1 \\ 1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1/2 & 1/2 & 0 & -1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} & 0 \\ -1/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & 1/2 & 0 & 1/\sqrt{2} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}.$$

Find $\hat{x}$ so that $\boxed{\text{minimize}_x \|Ax - b\|_2 = \|A\hat{x} - b\|_2}$ by exploiting the QR factorization.

3. Implement a Matlab routine `solvelsp.m` to solve the least squares problem

$$\text{minimize}_{x \in \mathbb{R}^n} \|Ax - b\|_2$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ $(m \geq n)$. Your Matlab routine must take two input parameters A and b. It must return one output parameter, the solution of the least squares problem $\hat{x} \in \mathbb{R}^n$ such that

$$\|A\hat{x} - b\|_2 = \text{minimize}_{x \in \mathbb{R}^n} \|Ax - b\|_2.$$

You can use the built-in Matlab routine `qr` to compute the QR factorization of A.

Test your implementation with two random matrix-vector pairs (A_1, b_1) and (A_2, b_2) where $A_1 \in \mathbb{R}^{4\times 3}$, $A_2 \in \mathbb{R}^{6\times 4}$, $b_1 \in \mathbb{R}^4$ and $b_2 \in \mathbb{R}^6$. Solve the least squares problems

$$\text{minimize}_{x \in \mathbb{R}^3} \|A_1 x - b_1\|_2 \quad \text{and} \quad \text{minimize}_{x \in \mathbb{R}^4} \|A_2 x - b_2\|_2$$

by calling your Matlab routine `solvelsp`. The Matlab command $A\backslash b$, when the system $Ax = b$ is inconsistent, returns the least squares solution $\hat{x}$. (Indeed it also first computes the QR factorization by Householder reflectors then exploits it.) Compare the results returned by your routine with the results returned by the built-in function in Matlab.

4. Solve parts **(a)** and **(b)** of questions **P8.4**, **P8.13** on page 325 (that is find the interpolating polynomials only in Lagrange and Newton forms) and question **A8.11** on page 331 in Fausett.

5.

(a) Estimate the integral

$$\int_{-1}^{1} e^{-x^3}\, dx$$

using the Trapezoidal rule, Simpson's rule and Gaussian quadrature with weight function $W(x) = 1$ and $n = 2$ (*i.e.* with three nodes x_0, x_1, x_2).

(b) Given the definite integrals

$$\int_{-1}^{1} x^2 + 3x + 4\, dx \quad \text{and} \quad \int_{-1}^{1} 3x^4 + 5x^3 - 2x^2 + x - 4\, dx.$$

Which of the quadrature formulas

(i) Trapezoidal rule

(ii) Simpson's rule

(iii) Gaussian quadrature with $W(x) = 1$ and $n = 2$ (*i.e.* with three nodes x_0, x_1, x_2).

would yield the exact value for each of these definite integrals? Answer the question without evaluating definite integrals and quadrature formulas.

6. Derive the Gaussian quadrature with the weight function $W(x) = \sqrt{\frac{1}{1-x^2}}$ and with two nodes (that is the function $f_n(x)$ interpolating the integrand $f(x)$ goes through $(x_0, f(x_0)), (x_1, f(x_1))$ for some distinct $x_0, x_1 \in [-1, 1]$ where x_0, x_1 are called the nodes of the quadrature formula). What is the degree of exactness of this quadrature formula for the integral

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \ ?$$