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Math 304 (Spring 2010)

Study Guide for Week 8
Homework 6 concerns the topics listed below.

- Derivatives of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$
- Generalized chain rule
- Linear approximations for multivariate functions
- Multivariate Newton's method (Fausett - 7.1, Quarteroni&Sacco&Saleri, 7.1.1)
- Convergence properties of Newton's method (Quarteroni&Sacco&Saleri, 7.1.1)

Homework 6 (due on April 26th, Monday by 4pm)
In Matlab question (questions **1(b)** and **6**) attach the Matlab outputs and print-outs of the Matlab routines that you implemented.

1. Consider the vector-valued function $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$F(x) = \begin{bmatrix} 2x_1 e^{x_3} - x_3^2 \sin x_1 \\ 2x_2 \\ x_1^2 e^{x_3} + 2x_3 \cos x_1 \end{bmatrix}$$

(a) Find $F'(x)$, the Jacobian of F.

(b) Given the point $x_k = (-1, 2, -2)^T$, find the linear approximation $L(x)$ at x_k such that

$$L(x_k) = F(x_k) \text{ and } L'(x_k) = F'(x_k).$$

Compute a zero of the function $L(x)$ using Matlab.

(c) Find the linear approximation about the point $x_k = (0, 0, 0)^T$.

2. Carry out one iteration of Newton's method applied to $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x) = \begin{bmatrix} x_1 + x_2 \\ (2x_1 - 1)^2 + (2x_2 - 1)^2 - \frac{2}{3} \end{bmatrix}$$

starting with the initial guess $x_0 = (1, -1)$.

3. Given the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x) = [x_1 \ \ x_2] \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [4 \ \ -2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and the vector $p = [1 \ \ -1]^T$. Let $\ell : \mathbb{R} \to \mathbb{R}$, $\ell(\alpha) = f(x + \alpha p)$ and $L : \mathbb{R} \to \mathbb{R}^2$, $L(\alpha) = \nabla f(x + \alpha p)$. Compute the derivatives, $\ell'(\alpha)$ and $L'(\alpha)$.

4. To compute $\sqrt{5}$ one approach is to apply Newton's method to the function $f(x) = x^2 - 5$.

(a) Is $f(x) = x^2 - 5$ Lipschitz continuous on $\mathcal{D} = [0, 2]$? Is the derivative $f'(x)$ Lipschitz continuous on $\mathcal{D} = [0, 2]$?

(b) Find the Newton update rule for $f(x) = x^2 - 5$, *i.e.* find the relation between x_{k+1} and x_k.

(c) Suppose that the sequence $\{x_k\}$ converges to $\sqrt{5}$. What is the rate of convergence that you would expect in theory?

5. Newton's method may not converge, when the initial guess is not sufficiently close to a solution. Consider for example the polynomial $f(x) = -x^7 + x^3 + 8x$. Show by hand that the basic Newton's method does not converge with the initial guess $x_0 = 1$.

6. Recall that a scalar λ satisfying $Ax = \lambda x$ for some nonzero $x \in \mathbb{R}^n$ is called an *eigenvalue* of A. The vector x is called an *eigenvector* associated with λ.

Define an iteration of Newton's method for a zero (which is a pair of eigenvalue and an associated unit eigenvector) of the $n+1$ equations

$$(A - \lambda I)x = 0, \quad x^T x = 1,$$

in the $n+1$ unknowns (x, λ). Use the m-file `Newton.m` provided on the course webpage (for the usage of `Newton.m` see question **5(e)** in Homework 5) to find an eigenvector and eigenvalue for the matrix

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{starting at } x_0 = \begin{bmatrix} 1/5 \\ -1/5 \\ 4/5 \\ 1 \end{bmatrix}.$$

Note that you need to implement an m-file computing the function $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$

$$f\left(\begin{bmatrix} x \\ \lambda \end{bmatrix}\right) = \begin{bmatrix} (A - \lambda I)x \\ x^T x - 1 \end{bmatrix}$$

and its Jacobian.