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Math 409/509 (Spring 2011)

Study Guide for Homework 6
This homework concerns the following topics.

- Sensitivity of constrained optimization problems and Lagrange multipliers (Nocedal and Wright, Sec 12.2)
- Primal-dual interior point methods especially for linear programs (Nocedal and Wright, Sec 14.1 and 14.4 for the analysis)
- Penalty function method for nonlinear programs (Nocedal and Wright, Sec 17.1)
- Logarithmic barrier method for nonlinear programs (Nocedal and Wright, Sec 17.2)

Homework 6 (will not be collected)

1. Consider the standard linear program (LP)

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^n} & c^T x \\ \text{subject} & Ax = b + \epsilon \\ & x \geq 0 \end{array} \qquad (1)$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $\epsilon \in \mathbb{R}^m$ is the perturbation vector.

View the optimal objective value $f_*(\epsilon)$ for the (LP) above as a function of ϵ. Find $\nabla f_*(\epsilon)$ at $\epsilon = 0$.

2. Consider the following dietary problem. Your doctor advises you to eat only cheese and fish in your diet. The amount of protein, calories are listed in the table below for each serving of cheese and fish as well as their costs.

	Cheese	Fish
Protein in gr	7.0	8.0
Calories in cal	105	90
Cost in $	8.0	11.0

Your protein and calorie intakes cannot exceed 10g and 150cal per day, respectively. Additionally your diet should include at least 0.5 serving of fish.

(a) Pose the problem of minimizing the total cost of your diet on a given day subject to the constraints listed above as a linear program in standard form.

(b) Write down the strictly feasible region with respect to both the primal and dual variables.

(c) Suppose you are given a strictly feasible point (x_k, π_k, s_k) and the centrality parameter μ_k. Write down one iteration of the pure Newton's method applied for the solution of the centrality conditions.

3. Consider the nonlinear program below.

$$\begin{array}{lll} \text{minimize}_{x\in\mathbb{R}^n} & f(x) & \\ \text{subject} & c_i(x) \geq 0 \;\; i = 1,\dots,m \\ & \tilde{c}_j(x) = 0 \;\; j = 1,\dots,\ell \end{array} \tag{2}$$

(a) Write down the *centrality conditions* for the nonlinear program above similar to what we did in class for linear programs.

(b) Derive one iteration of the Newton's method for the solution of the centrality conditions in part **(a)**. (Note: assume centrality parameter is fixed and does not depend on primal variables as well as dual variables associated with inequality constraints.)

4. This question concerns the following hanging chain problem. Suppose you have two bars with lengths $L_1 = 6$ and $L_2 = 7$. The left end point of the first bar is fixed at $(x_0, y_0) = (0, 0)$, whereas the right end point of the second bar is fixed at $(x_2, y_2) = (4, 5)$. The coordinates (x_1, y_1) of the joint connecting the right end of the first bar to the left end of the second bar are variables. (You can assume all units are in meters.)

Your task is to minimize the total potential energy of the bars subject to the constraints that

- both of the bars are above the floor whose (x, y) coordinates are related by $y = x - 2$, and
- the distance between the end points of each bar must match the actual length of the bar.

The potential energy of the jth bar is given by

$$L_j \frac{y_j + y_{j-1}}{2}.$$

(a) Pose the problem of minimizing the total potential energy of the bars subject to the constraints above as a nonlinear program.

(b) Write down the mixed penalty and logarithmic barrier functions associated with the nonlinear program in part **(a)**.

5. Consider the penalty function

$$P(x, \mu) = f(x) + \frac{1}{2\mu} \sum_{j=1}^{\ell} \tilde{c}_j^2(x)$$

for the nonlinear optimization problem

$$\begin{array}{ll} \text{minimize}_{x\in\mathbb{R}^n} & f(x) \\ \text{subject} & \tilde{c}_j(x) = 0 \;\; j = 1,\dots,\ell \end{array}$$

with equality constraints (NEP) only.

(a) Derive one iteration of the Pure Newton's method applied to $P(x, \mu)$ to find a local minimizer for a fixed μ.

(b) Consider a continuous sequence $\{x(\mu)\}$ of local minimizers for the penalty function $P(x, \mu)$ such that

$$\lim_{\mu \to 0^+} x(\mu) = x_*$$

satisfies the KKT condition for (NEP). It can be shown that

$$\lim_{\mu \to 0^+} -\frac{\tilde{c}(x)}{\mu} = \lambda_*$$

where λ_* is the Lagrange variable associated with x_* in the KKT conditions at x_*.

Show that

$$\lim_{\mu \to 0^+} \|\nabla P_{xx}^2(x(\mu), \mu)\| = \infty.$$

Above ∇P_{xx}^2 denotes the Hessian of P with respect to x. (Note: The practical importance of this result is that it becomes more and more difficult to solve the linear systems resulting from Newton's method numerically as $\mu \to 0^+$.)

6. [Nocedal and Wright - 17.2, p. 526] Consider the scalar minimization problem

$$\text{minimize}_x \frac{1}{1+x^2}, \quad \text{subject to } x \geq 1.$$

Write down $P(x, \mu)$ for this problem, and show that $P(x, \mu)$ is unbounded below for any positive value of μ.

7. [Nocedal and Wright - 17.3, p. 527] Consider the scalar minimization problem

$$\text{minimize } x, \quad \text{subject to } x^2 \geq 0, \; x + 1 \geq 0,$$

for which the solution is $x^* = -1$. Write down $P(x, \mu)$ for this problem and find its local minimizers. Show that for any nonnegative sequence $\{\mu_k\}$, such that $\mu_k \to 0$ as $k \to \infty$, there exists a corresponding sequence of local minimizers $x(\mu_k)$ that converges to 0.