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Math 409/509 (Spring 2011)

Study Guide for Homework 5
This homework concerns the nonlinear optimization problems with inequality constraints, and nonlinear programs, more specifically the topics listed below. Please don't hesitate to ask for help if any of these topics is unclear. Sections 5.1 and 5.2 in Gill&Wright are relevant.

- Active normal cone for inequality constraints
- Farka's lemma
- Constraint qualification for inequality constraints
- First order optimality condition for inequality constraints
- Complementarity conditions
- First order optimality conditions (also known as Karush-Kuhn-Tucker or KKT conditions) for nonlinear programs
- Dual problem for a linear program

Homework 5 (due on May 30th, Monday by 17:00)

1. Consider the nonlinear inequality constrained program (NIP)

$$\begin{array}{ll} \text{minimize}_{x\in\mathbb{R}^2} & -x_1 + x_2 \\ \text{subject} & -x_1^3 - x_2 \geq 0 \\ & x_1 - x_2^2 \geq 0 \end{array} \qquad (1)$$

(a) Draw the feasible region.

(b) Find the tangent cones at $\bar{x} = (1, -1)$ and $\hat{x} = (\frac{1}{2}, -\frac{1}{8})$.

(c) Is the constraint qualification satisfied at $\bar{x}$ and $\hat{x}$?

(d) The unique minimizer of the problem is $\bar{x}$. Plot the active normal cone at $\bar{x}$. Verify geometrically and algebraically that the gradient of the objective function is contained in the active normal cone.

(e) Plot the active normal cone at $\hat{x}$ and verify that $\hat{x}$ is not a minimizer.

(f) Write down a direction p such that $\hat{x}+\alpha p$ for all $\alpha > 0$ sufficiently small satisfies the following conditions.

(i) $f(\hat{x} + \alpha p) < f(\hat{x})$

(ii) $(\hat{x} + \alpha p) \in \mathbb{F}$

Here $f(x) = -x_1 + x_2$ denotes the objective function, whereas $\mathbb{F}$ denotes the feasible region. The direction p that you determine is a feasible descent direction from $\hat{x}$.

2. Consider the inequality constrained optimization problem

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^2} & -\frac{16}{3}x_1^3 + x_2 \\ \text{subject} & -x_1 - x_2^2 \geq 0 \\ & x_1 + 4 \geq 0. \end{array} \tag{2}$$

(a) Draw the feasible region and the tangent cone at $\bar{x} = (-4, 2)$. Does the constraint qualification hold at $\bar{x}$?

(b) Plot the active normal cone at $\hat{x} = (-\frac{1}{4}, -\frac{1}{2})$. Does $\hat{x}$ satisfy the first-order optimality conditions?

3.[Nocedal and Wright - 12.19, p.354] Consider the problem

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^2} & -2x_1 + x_2 \\ \text{subject} & (1 - x_1)^3 - x_2 \geq 0 \\ & 0.25x_1^2 + x_2 - 1 \geq 0 \end{array} \tag{3}$$

The optimal solution is $x_* = (0, 1)$, where both constraints are active. Does the linear independence constraint qualification hold at x_*? Are the first-order optimality conditions satisfied?

4. Show that the set (called the active normal cone)

$$\mathcal{N}_a(x) = \{J_a(x)\lambda : \lambda \geq 0\}$$

is a *cone* for all x. Above $J_a(x)$ denotes the Jacobian of the active constraints at x.

5. Consider the minimization of a nonlinear function subject to linear inequality constraints, that is consider the problem

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^n} & f(x) \\ \text{subject} & Ax \geq b \end{array} \tag{4}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a twice-continuously differentiable function, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Prove that the constraint qualification holds at all x for this problem. You need to establish the equivalence of the tangent cone $T^0(x)$, and the set

$$\mathcal{S}_a(x) = \{p : J_a(x)p \geq 0\}$$

at all x where $J_a(x)$ denotes the Jacobian of the active constraints at x.

6. [Nocedal and Wright - 12.4, p. 358] Let $v_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$ be twice-continuously differentiable functions and

$$f(x) = \max_{j=1,\ldots,m} v_j(x).$$

Consider the unconstrained optimization problem

$$\text{minimize}_{x \in \mathbb{R}^n} f(x). \tag{5}$$

(a) Argue why $f(x)$ is not differentiable. Specifically specify the points where $f(x)$ is not differentiable. (Note: this means that you cannot use the methods we have seen for unconstrained optimization, since they are devised for smooth functions only.)

(b) Pose (5) as a smooth constrained optimization problem.

7. [Nocedal and Wright - 12.15, p. 359] Consider the halfspace defined by

$$\mathcal{H} = \{x \in \mathbb{R}^n : a^T x + \alpha \geq 0\}$$

where $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ are given. Formulate and solve the optimization problem for finding the point x in $\mathcal{H}$ that has the smallest Euclidean norm.

8. Write down the Karush-Kuhn-Tucker conditions for the following problem in complementarity form.

$$\begin{aligned} \text{minimize}_{x \in \mathbb{R}^n} \quad & x^T H_1 x + g_1^T x \\ \text{subject} \quad & x^T H_2 x + g_2^T x = b \\ & x_j \in [\ell_j, u_j], \; j = 1, \ldots, n \end{aligned} \tag{6}$$

Above $H_1, H_2 \in \mathbb{R}^{n \times n}$, $g_1, g_2 \in \mathbb{R}^n$ and $b, \ell_j, u_j \in \mathbb{R}$ for $j = 1, \ldots, n$.

9. Consider the standard linear program (LP)

$$\begin{array}{lrcl} \text{minimize} & 2x_1 \quad\quad + x_3 + x_4 & & \\ \text{subject to} & x_1 + x_2 + x_3 \quad\quad & = & 1 \\ & x_2 + x_3 + x_4 & = & 2 \\ & x_1 & \geq & 0 \\ & x_2 & \geq & 0 \\ & x_3 & \geq & 0 \\ & x_4 & \geq & 0. \end{array}$$

(a) Write down the dual problem for the LP above.

(b) Express the dual problem in part **(a)** as a standard LP of the form

$$\begin{aligned} \text{minimize}_{\pi} \quad & c^T \pi \\ \text{subject} \quad & A\pi = b \\ & \pi \geq 0 \end{aligned} \tag{7}$$

A point π_* must be a maximizer of the dual problem in part **(a)** if and only it is a minimizer of the standard LP (7).