SOLUTIONS OF HOMEWORK IV

1- (a) $- x(0) = (\sin 0, \cos 0 - 1) = (0,0)$ ✓

$- x'(0) = (\cos 0, -\sin 0) = (1,0) \neq (0,0)$ ✓

$- c(x(\alpha)) = \sin^2\alpha + \cos^2\alpha - 1 = 0$ ✓

So, $x(\alpha)$ is a feasible path at $\bar{x} = (0,0)$

(b) $T^0 = \{ x'(0) : x(\alpha)$ is a feasible path at $\bar{x} = (0,0) \}$

is tangent cone at $\bar{x}$.

We know that $c(x(\alpha)) = 0$, say $x_1 = \pm\alpha$ then

$1 - \alpha^2 = (x_2+1)^2 \Rightarrow x_2 + 1 = \sqrt{1-\alpha^2} \Rightarrow x_2 = \sqrt{1-\alpha^2} - 1$

then $x_1(\alpha) = (\alpha, \sqrt{1-\alpha^2} - 1)$ and $x_2(\alpha) = (-\alpha, \sqrt{1-\alpha^2} - 1)$

are 2 feasible paths-

$$x_1'(\alpha) = \left(1, \frac{-\alpha}{\sqrt{1-\alpha^2}}\right) \; ; \; x_2'(\alpha) = \left(-1, \frac{-\alpha}{\sqrt{1-\alpha^2}}\right)$$

$\Rightarrow x_1'(0) = (1, 0)$, $x_2'(0) = (-1, 0)$

We know that if $x'(0) \in T^0$ then $c\,x'(0) \in T^0$

for $c \geq 0$

So, $T^0 = \left\{ c \begin{bmatrix} 1 \\ 0 \end{bmatrix} : c \in \mathbb{R} \right\}$

$J(\bar{x}) = [\, 2\bar{x}_1 \quad 2(\bar{x}_2+1) \,] \Rightarrow \text{Null}(J(\bar{x})) = \{ p \in \mathbb{R}^2 : J(\bar{x})\, p = 0 \}$

$$[\, 2\bar{x}_1 \quad 2(\bar{x}_2+1) \,] \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = [\, 0 \quad 1 \,] \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0 \Rightarrow p = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where $c \in \mathbb{R}$, Since $T^0 = \text{Null}\, J(\bar{x})$ at $\bar{x} = (0,0)$

Therefore constraint qualification satisfied at $\bar{x}$.

c) $f(x) = 3x_2 + x_1^2 + x_2^2 \Rightarrow f(x(\alpha)) = \sin^2\alpha + (\cos\alpha - 1)^2 + 3(\cos\alpha - 1)$

$= \underbrace{\sin^2\alpha + \cos^2\alpha}_{1} - 2\cos\alpha + 1 + 3\cos\alpha - 3$

$= \cos\alpha - 1 \Rightarrow f(x(0)) = \cos 0 - 1 = 0$

d) $J(x) = [\,2x_1 \quad 2x_2+2\,]$, $L(x,\lambda) = f(x) - \lambda^T c(x)$

$= 3x_2 + x_1^2 + x_2^2 - \lambda(x_1^2 + (x_2+1)^2 - 1)$

$$\nabla \mathcal{L}(x,\lambda) = \begin{bmatrix} \nabla f(x) - J(x)^T\lambda \\ -c(x) \end{bmatrix} = \begin{bmatrix} 2x_1 - 2\lambda x_1 \\ 3 + 2x_2 - 2\lambda x_2 - 2\lambda \\ 1 - x_1^2 - (x_2+1)^2 \end{bmatrix}$$

$$\nabla^2 \mathcal{L}(x,\lambda) = \begin{bmatrix} \nabla^2 f(x) - \lambda\nabla^2 c(x) & -J^T(x) \\ -J(x) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-2\lambda & 0 & -2x_1 \\ 0 & 2-2\lambda & -2x_2-2 \\ -2x_1 & -2x_2-2 & 0 \end{bmatrix}$$

e) $\nabla \mathcal{L}(x,\lambda) = 0 \Rightarrow 2x_1 = 2\lambda x_1 \Rightarrow$ either $\lambda = 1$ or $x_1 = 0$

Let $x_1 = 0$ then $1 - 0^2 - (x_2+1)^2 = 0 \Rightarrow x_2 = 0$ or $x_2 = -2$

$\boxed{\text{for } x_2 = 0}$

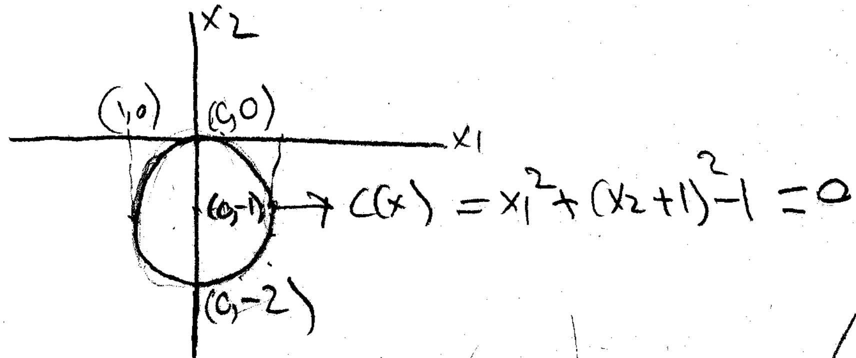
$\Rightarrow 3 + 0 - 0 - 2\lambda = 0 \Rightarrow \lambda = \frac{3}{2}$; $3 - 4 + 4\lambda - 2\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$

Let $\lambda = 1$ then $3 + 2x_2 - 2x_2 - 2\lambda = 0 \Rightarrow \lambda = \frac{3}{2}$ contradiction

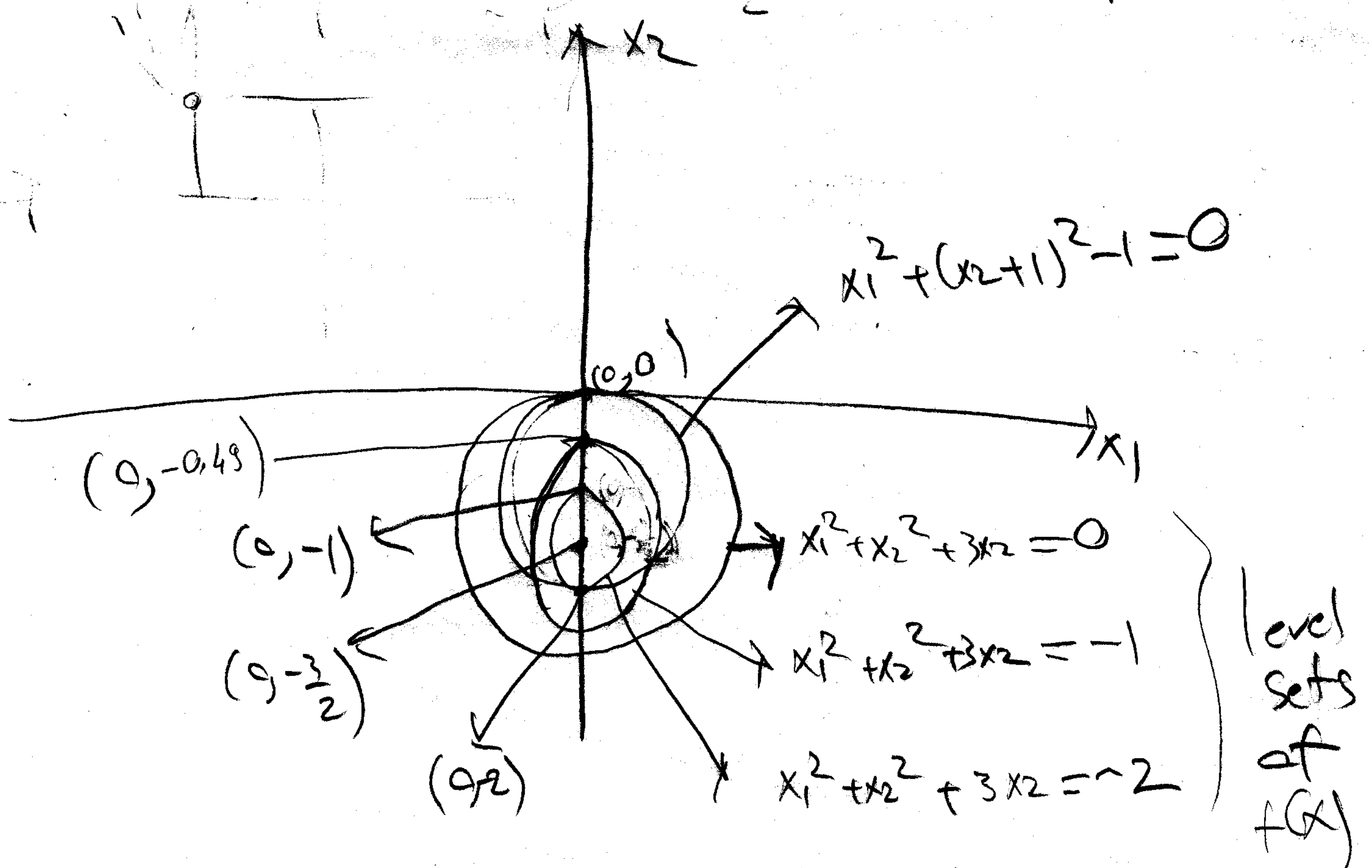
So there are 2 possibility for constraint minimizer either $(0,0)$ or $(0,-2)$,

We know that constraint qualification satisfied at $(0,0)$

From First order Optimality Conditions, we can say $(0,0)$ is a candidate for local minimizer but it is not certain (Since First order opt. conds. are necessary but not sufficient)



We can deduce that $(0,0)$ is not a minimizer since $f(0,0) = 0 > f(0,-2) = -2$



From level sets we can deduce that $(0,-2)$ is constrained minimizer of f.

2- a) If x^* is a local minimizer where constraint qualification holds, then $\exists$ a $\lambda \in \mathbb{R}^m$ s.t.

(i) $\nabla f(x^*) = J(x^*)^T \lambda$

(ii) $c(x^*) = 0$

b) $\mathcal{L}(x,\lambda) = (x_1-1)^2 + x_2^2 + \lambda(x_1 - x_2^2)$

i) $J(x) = (\nabla c(x))^T = [-1 \quad -2x_2]$

$$\nabla \mathcal{L}(x,\lambda) = \begin{bmatrix} 2x_1 - 2 + \lambda \\ 2x_2 - 2x_2\lambda \\ x_1 - x_2^2 \end{bmatrix}$$

$$\nabla^2 \mathcal{L}(x,\lambda) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2-2\lambda & -2x_2 \\ 1 & -2x_2 & 0 \end{bmatrix}$$

ii) $(\frac{1}{2}, \frac{1}{\sqrt{2}}, 1)$ $(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 1)$ $(0, 0, 2)$ are the stationary points of $\mathcal{L}$ satisfying first order optimality conditions

3-

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function [c,J] = constraint(x1,x2)

c = x1 + x2 - x1 * x2 - 3/2;
J = [1-x2  1-x1];

return
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At $x_1 = (0.1, -0.5)$
$c(x) = -1.85$
$J(x) = [1.5 \quad 0.9]$

At $x_2 = (0.5, -1)^T$
$c(x) = -1.5$ $\quad J(x)\ [2 \quad 0.5]$

At $x_3 = (1.18249728, -1.73976692)^T$
$c(x) = 1.0734 \times 10^{-8}$ $\quad J(x) = [2.7398 \quad -0.18125]$

for x_1 and x_2, $c(x_1) \neq 0$ and $c(x_2) \neq 0$

So x_1 and x_2 are not in the feasible set.

$c(x_3) \simeq 0$, If we put a tolerance ε, then

then we can set $c(x_3) = 0$ when $|c(x_3)| \leq \varepsilon$

So, we can claim that x_3 is a candidate for local minimizer if $\exists \lambda \in \mathbb{R}$ s.t $\nabla f(x_3) = \lambda J^T(x_3)$

4- minimize$_{x \in \mathbb{R}^n}$ $\frac{1}{2} x^T G x + d^T x$

subject $Ax = b$

where $G \in \mathbb{R}^{n \times n}$ is symmetric, $A \in \mathbb{R}^{m \times n}$, $d \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

a) ~~Since~~ constraint is linear, constraint qualification is hold.

x^* is local minimizer

$\Rightarrow$

(i) $c(x^*) = 0 \Rightarrow Ax^* = b$

(ii) $\nabla f(x^*) = Gx + d = J(x^*)^T \lambda = A^T \lambda$, $\exists \lambda \in \mathbb{R}^m$

b) Since the constraint is linear, constraint qualification holds. So $T^\circ(\bar{x}) = \text{Null}(J(\bar{x}))$

$\forall \bar{x} \in \mathbb{R}^n$, $J(\bar{x}) = A$, Therefore $T^\circ(\bar{x}) = \text{Null}(A)$.

5-

a) $L(x,\lambda) = f(x) - \lambda^T c(x) = x^T x - \lambda^T (x^T H x - 1)$

and $J(x) = 2x^T H$, where $H_{n \times n}$ is symmetric

$$\nabla L(x,\lambda) = \begin{bmatrix} \nabla f(x) - J(x)^T \lambda \\ -c(x) \end{bmatrix} = \begin{bmatrix} 2x - 2(Hx)\lambda \\ 1 - x^T H x \end{bmatrix}$$

b) $Hv = \mu v$, $\nabla L\left(\frac{v}{\sqrt{\mu}}, \frac{1}{\mu}\right) = \begin{bmatrix} \frac{2v}{\sqrt{\mu}} - 2\frac{Hv}{\sqrt{\mu}}\frac{1}{\mu} \\ 1 - \frac{v^T}{\sqrt{\mu}}\frac{Hv}{\sqrt{\mu}} \end{bmatrix}$

$$= \begin{bmatrix} \frac{2v}{\sqrt{\mu}} - \frac{2\mu v}{\sqrt{\mu}}\frac{1}{\mu} \\ 1 - \frac{v^T}{\sqrt{\mu}} \cdot \frac{\mu v}{\sqrt{\mu}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 - v^T v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since $\|v\| = 1$

Therefore $\left(\frac{v}{\sqrt{\mu}}, \frac{1}{\mu}\right)$ is a stationary point of $L(x,\lambda)$

c) $$\nabla^2 L(x,\lambda) = \begin{bmatrix} 2I - 2H\lambda & -2Hx \\ -2x^T H & 0 \end{bmatrix}$$

We seek a search direction $\Delta = \begin{bmatrix} \Delta x_k \\ \Delta \lambda_k \end{bmatrix}$ such that

$$\begin{bmatrix} 2I - 2H\lambda_k & -2Hx_k \\ -2x_k^T H & 0 \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta \lambda_k \end{bmatrix} = \begin{bmatrix} 2x_k - 2(Hx_k)\lambda_k \\ 1 - x_k^T H x_k \end{bmatrix}$$

6 —

a) $L(x,\lambda) = x^T H x - \lambda^T (x^T x - 1)$

$$\nabla L(x,\lambda) = \begin{bmatrix} 2Hx - 2x\lambda \\ 1 - x^T x \end{bmatrix}, \quad \nabla L(v,\mu) = \begin{bmatrix} 2Hv - 2v\mu \\ 1 - v^T v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) $F'(x,\lambda) = (\nabla L(x,\lambda))' = \nabla^2 L(x,\lambda)$

$$= \begin{bmatrix} 2H - 2\lambda I & -2x \\ -2x^T & 0 \end{bmatrix}$$

c)

Let $\nabla L(x,\lambda) = \begin{bmatrix} 2Hx - 2x\lambda \\ 1 - \|x\|^2 \end{bmatrix} = 0$

Then $\|x\| = 1$ (i.e. x is unit vector) and $Hx = \lambda x$

(i.e. x is an eigenvector of H with eigenvalue λ)

We know that Method of multipliers guarentee to converge only to a stationary point of $\mathcal{L}(x,\lambda)$

Therefore, given an estimate (x_k, λ_k) for minimizer and Lagrange multiplier,

$$\begin{bmatrix} 2H - 2\lambda_k I & -2x_k \\ -2x_k^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta \lambda_k \end{bmatrix} = \begin{bmatrix} 2Hx - 2x\lambda \\ 1 - x^T x \end{bmatrix}$$

and $\begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} + \alpha_k \begin{bmatrix} \Delta x_k \\ \Delta \lambda_k \end{bmatrix}$ (determined by inexact line search), then $\lim_{k \to \infty} \begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} = \begin{bmatrix} x \\ \lambda \end{bmatrix}$, where x is a unit eigenvector of H with e-value λ

7-

a) $\mathcal{L}(x,\lambda) = e^{x_1}(4x_1^2+2x_2^2+4x_1x_2+2x_2+1) - \lambda(x_1+x_2-x_1x_2-\frac{3}{2})$

$$\nabla \mathcal{L}(x,\lambda) = \begin{bmatrix} e^{x_1}(4x_1^2+2x_2^2+4x_1x_2+2x_2+1+8x_1+4x_2) - \lambda + x_2 \\ e^{x_1}(4x_2+4x_1+2) - \lambda + x_1 \\ -x_1-x_2+x_1x_2+\frac{3}{2} \end{bmatrix}_{3\times 1}$$

$$\nabla^2 \mathcal{L}(x,\lambda) = \begin{bmatrix} e^{x_1}(4x_1^2+2x_2^2+4x_1x_2+2x_2+16x_1+8x_2+9) & (e^{x_1}(4x_1+4x_2+6)+1) & -1 \\ (e^{x_1}(4x_2+4x_1+6)+1) & 4e^{x_1} & -1 \\ -1+x_2 & -1+x_1 & 0 \end{bmatrix}_{3\times 3}$$

b) Matlab

c) Matlab