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Math 304 (Spring 2010)

Study Guide for Week 5
Homework 4 concerns the topics listed below.

- Definitions of eigenvalues, eigenvectors and eigenspaces (Watkins - 5.2)
- Basic facts concerning eigenvalues and eigenvectors; some of these are listed below (Watkins - 5.2)
  - The eigenvalues of A are the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
  - The eigenvectors of A associated with an eigenvalue λ are non-zero vectors in the null-space of $A - \lambda I$.
- Similarity transformations in the form $A \to B := T^{-1}AT$. Here B and A are said to be similar. Watkins - 5.4)
  - A similarity transformation preserves the eigenvalues.
- Power iteration (Watkins - 5.3, Fausett - 5.1)
- Rate of convergence

Homework 4 (due on March 26th, Friday by 4pm)
In Matlab question (questions **3**) attach the Matlab outputs and print-outs of the Matlab routines that you implemented.

1. Perform all calculations by hand in this question.

(a) For each of the matrices given below write down the characteristic polynomial and calculate the eigenvalues.

$$A_1 = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) For the matrix A_2 in part **(a)**

(i) Calculate an eigenvector associated with each eigenvalue.

(ii) Provide the eigenspace associated with each eigenvalue.

(c) Suppose that you apply the power iteration to A_1 and A_2. Which eigenvalue and eigenvector would you expect the power iteration to converge for each of the matrices A_1 and A_2?

2. This question concerns similarity transformations.

(a) Which of the following transformations would necessarily preserve the eigenvalues of $A \in \mathbb{R}^{n \times n}$? (*i.e.* A transformation $A \to B$ preserves eigenvalues if and only if A and B have the same set of eigenvalues.) Explain why or why not?

(i) The transformation $A \to B := SAS$ where $S \in \mathbb{R}^{n \times n}$ is invertible

(ii) The transformation $A \to B := SAS^{-1}$ where $S \in \mathbb{R}^{n \times n}$ is invertible

(iii) The transformation $A \to B := QA$ where $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix

(iv) The transformation $A \to B := Q^T AQ$ where $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix

(b) Recall that if $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ are similar, then A and B have the same set of eigenvalues. The converse is not true in general. Write down two 2×2 matrices that share the same set of eigenvalues, but are not similar.

3. Implement a Matlab routine `power_iter.m` to compute an eigenvalue λ and an associated eigenvector v of an $n \times n$ matrix A by power iteration, respectively.

The routine must return two output arguments, the computed eigenvalue λ and the associated eigenvector v. It must take one input argument, the $n \times n$ matrix A for which an eigenvalue and an eigenvector are sought. You can choose a randomly generated vector (by typing the command `randn(n,1)`) as your initial estimate $q_0 \in \mathbb{C}^n$ for the eigenvector. Display the estimate of the eigenvalue at each iteration (by typing `display(q'*A*q)` where `q` is the current estimate of the eigenvector with 2-norm equal to one) so that you can observe the rate of convergence. You should terminate when the eigenvector estimates q_k and q_{k+1} at two consecutive iterations are sufficently close to each other, *e.g.* $\|q_k - q_{k+1}\| \leq 10^{-15}$. Note that in your Matlab implementation you will need a while loop (instead of a for loop) in the following form

```
while norm(q - qold) > 10^-15
   ...
end
```

where `q`, `qold` correspond to the eigenvector estimates at the current and previous iterations. (Initially you can set `qold = zeros(n,1)` before the while loop to make sure that your program iterates at least once.)

Test your implementation with the following matrices.

$$B_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -0.8 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 & 1/10 \\ 0 & -0.8 & 0 \\ 1/10 & 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 0 & 1/100 \\ 0 & -0.8 & 0 \\ 1/100 & 0 & 1 \end{bmatrix}$$

(a) Which eigenvalue and eigenvector does the power iteration converge in each case? Is this expected in theory? Explain.

(b) For which of the matrices above the convergence is fastest, for which the convergence is slowest? Try to explain why this is expected in theory.