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Math 409/509 (Spring 2011)

Study Guide for Homework 4
This homework concerns the nonlinear optimization problems with equality constraints, more specifically the topics listed below. Please don't hesitate to ask for help if any of these topics is unclear. Sections 4.1, 4.2 and 4.4 in Gill&Wright are relevant.

- Feasible path and tangent cone
- Constraint qualification
- First order necessary conditions
- Lagrangian function
- Method of multipliers

Homework 4 (due on May 4th, Wednesday by 14:00)
Question **3** and **7** requires computations in Matlab. Attach the print-outs of the m-files that you implemented and Matlab outputs.

1. Consider the nonlinearly constrained problem

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^2} & 3x_2 + x_1^2 + x_2^2 \\ \text{subject} & x_1^2 + (x_2+1)^2 - 1 = 0. \end{array} \tag{1}$$

(a) Show that $x(\alpha) = (\sin\alpha, \cos\alpha - 1)^T$ is a feasible path for the nonlinear constraint $x_1^2 + (x_2 + 1)^2 - 1 = 0$ of problem (1).

(b) Compute the tangent cone and the null space of the Jacobian at $\bar{x} = (0,0)^T$. Is the constraint qualification satisfied at $\bar{x}$.

(c) If $f(x)$ denotes the objective function of problem (1), find an expression for $f(x(\alpha))$ and compute $f(x(0))$.

(d) Define the *Lagrangian function* $L(x, \lambda)$ and *constraint Jacobian* $J(x)$ for problem (1). Derive $\nabla L(x, \lambda)$, the gradient of the Lagrangian, and $\nabla^2 L(x, \lambda)$, the Hessian of the Lagrangian.

(e) Determine whether or not problem (1) has a constrained minimizer.

2. Let $f(x)$ and $c(x)$ be twice-continuously differentiable functions such that $f : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^m$. Consider the nonlinearly constrained problem

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^n} & f(x) \\ \text{subject} & c(x) = 0. \end{array}$$

(a) What are the first-order necessary conditions for x_* to be a local solution of this problem?

(b) Consider the specific problem

$$\begin{aligned} &\text{minimize}_{x\in\mathbb{R}^2} \quad (x_1-1)^2 + x_2^2 \\ &\text{subject} \qquad -x_1 + x_2^2 = 0. \end{aligned} \tag{2}$$

(i) Define the *Lagrangian function* $L(x, \lambda)$ and *constraint Jacobian* $J(x)$ for problem (2). Derive $\nabla L(x, \lambda)$, the gradient of the Lagrangian, and $\nabla^2 L(x, \lambda)$, the Hessian of the Lagrangian.

(ii) Find the points satisfying the first order optimality conditions.

3. Write a MATLAB function that will compute $c(x)$ and $J(x)$ for the constraint function

$$c(x) = x_1 + x_2 - x_1 x_2 - \frac{3}{2}.$$

Use your function to compute $c(x)$ and $J(x)$ at $x = (.1, -.5)^T$, $x = (.5, -1)^T$ and $x = (1.18249728, -1.73976692)^T$.

At each of these points, discuss the optimality of the constrained minimization problem

$$\begin{aligned} &\text{minimize}_{x\in\mathbb{R}^2} \quad e^{x_1}(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1) \\ &\text{subject} \qquad x_1 + x_2 - x_1x_2 - \tfrac{3}{2} = 0. \end{aligned}$$

based on the first-order optimality conditions.

4. Consider the equality constrained problem

$$\begin{aligned} &\text{minimize}_{x\in\mathbf{R}^n} \quad \tfrac{1}{2}x^T G x + d^T x \\ &\text{subject} \qquad Ax = b. \end{aligned} \tag{3}$$

where $G \in \mathbb{R}^{n\times n}$ is symmetric, $A \in \mathbb{R}^{m\times n}$, $d \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

(a) Write down the first order necessary conditions that a minimizer x_* of (3) must satisfy.

(b) Express the tangent cone $T^0(\bar{x})$ at any given point $\bar{x} \in \mathbb{R}^n$ for the equality constrained problem (3) in terms of the matrix A.

5. Consider the constrained optimization problem

$$\begin{aligned} &\text{minimize}_{x\in\mathbb{R}^n} \quad x^T x \\ &\text{subject} \qquad x^T H x - 1 = 0, \end{aligned} \tag{4}$$

where H is a symmetric $n \times n$ matrix. This problem can geometrically be interpreted as finding the point on an ellipsoid closest to the origin.

(a) Write down the Lagrangian function for (5) and calculate its gradient.

(b) Suppose $\mu \neq 0$ is an eigenvalue of H and v is an associated eigenvector of unit length. Show that $(\frac{v}{\sqrt{\mu}}, \frac{1}{\mu})$ is a stationary point of the Lagrangian function.

(c) The method of multipliers is the Newton's method applied to the Lagrangian function. Given $x_k \in \mathbb{R}^n$, an estimate for the minimizer, and λ_k, an estimate for the optimal Lagrange multiplier. Derive the linear system that needs to be solved to generate the search direction for the method of multipliers at (x_k, λ_k) for problem (5). The coefficient matrix and right-hand side vector of the linear system must depend on x_k, λ_k and H only.

6. Consider the constrained optimization problem

$$\begin{array}{ll} \text{minimize}_{x \in \mathbf{R}^n} & x^T H x \\ \text{subject} & x^T x - 1 = 0, \end{array} \tag{5}$$

where H is a symmetric $n \times n$ matrix.

(a) Prove that if v is a eigenvector of H with unit norm, and μ is its corresponding eigenvalue, then (v, μ) is a stationary point of the Lagrangian function $L(x, \lambda)$ for the problem (5).

(b) Derive an expression for the $(n+1)$-vector $\nabla L(x, \lambda)$. If $F(x, \lambda)$ denotes the function $\nabla L(x, \lambda)$, derive $F'(x, \lambda)$.

(c) Given an estimate (x, λ) of the solution and Lagrange multiplier, derive the specific form of the Newton equations associated with the method of multipliers (i.e., write down these equations in terms of x, H, etc.). Hence define an iterative method for finding an eigenvalue and eigenvector of a symmetric matrix H.

7. A completed m-file `Newton_backtrack.m` has been posted on the course webpage. Copy the file into your working directory. This question asks you to apply the method of multipliers to the equality-constrained problem in parts **(a)** and **(c)** below.

(a) For the constrained minimization problem

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^2} & e^{x_1}(4x_1^2 + 2x_2^2 + 4x_1 x_2 + 2x_2 + 1) \\ \text{subject} & x_1 + x_2 - x_1 x_2 - \frac{3}{2} = 0, \end{array}$$

calculate the gradient and Hessian of the Lagrangian function.

(b) Recall that if a point x_* is a local minimizer of a nonlinear equality constrained program (NEP), then there exists a λ_* such that (x_*, λ_*) is a stationary point of the Lagrangian function. The method of multipliers for (NEP) is Newton's method applied to the gradient of the Lagrangian function to locate a stationary point of the Lagrangian function. Use `Newton_backtrack.m` to solve the constrained minimization problem of part **(a)** starting at $x_0 = (2, \frac{1}{2})^T$, $\lambda_0 = 0$. *Do not alter the function* `Newton_backtrack.m` *in any way.*

(c) Repeat part (b), but change the constraint to $4x_1 - x_2 - 6 = 0$.

(d) Repeat part (c), but start at $x_0 = (1, -2)^T$,