SOLUTIONS OF HOMEWORK II

1) $f(x) = (x^2+1)(x+1) \qquad g(x) = (x-1)^2$

a) Newton Method for f. $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{(x_k^2+1)(x_k-1)}{3x_k^2-2x_k+1}$

for g = $x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)} = x_k - \frac{(x_k-1)^2}{2(x_k-1)}$

Secant Method for f: $x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k - f(x_{k-1})}$

$$= x_k - \frac{(x_k^2+1)(x_k+1)\ (x_k - x_{k-1})}{x_k^2+1(x_k-1) - (x_{k-1}^2+1)(x_{k-1}-1)}$$

for g: $x_{k+1} = x_k - \frac{g(x_k)\ (x_k - x_{k-1})}{g(x_k) - g(x_{k-1})} = x_k - \frac{(x_k-1)^2(x_k - x_{k-1})}{(x_k-1)^2 - (x_{k-1}-1)^2}$

b) for g: $g'(x_*) = 2(x_*-1) = 0 \Rightarrow$ q-rate convergence is linear

$$\Rightarrow \lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = \lim_{k\to\infty} \frac{\left| x_k - \frac{(x_k-1)^2}{2(x_k-1)} - 1 \right|}{|x_k - 1|} = 1/2$$

for f: $f'(x_*) = 3x_*^2 - 2x_* + 1 = 2 \neq 0$ q-rate convergence is quadratic

$$\Rightarrow \lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = \lim_{k\to\infty} \frac{\left| \frac{2x_k(x_k-1)^2}{3x_k^2-2x_k+1} \right|}{|x_k - 1|^2} = \lim_{k\to\infty} \frac{|2x_k|}{|3x_k^2-2x_k+1|} = 2$$

2 is finite : ⇒ quadratic convergence follows.

c)
```
function [f,g] = fun (x)

f = (x^2 +1) * (x-1)

g = 3 * x^2 - 2 * x +1;

return
```
} fun.m file

```
function [f,g] = g (x)
f = (x-1) ^ 2;
g = 2 * (x -1);

return
```
} g.m file

d, e) MATLAB

2) a) $\frac{1}{2} x^T A x + b^T x$, $f' = x^T A + b^T$

$l(x) = f(x_0) + f'(x_0)(x - x_0)$ where $x_0 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

since $x_0 \notin \mathbb{R}^3$ then we can't find the linear approximation

$\Rightarrow$ say $x_0 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ then , $x_0 \in \mathbb{R}^3$

$l(x) = \frac{17}{2} + [-3 \ -4 \ -5](x - x_0) = \frac{7}{2} + [-3 \ -4 \ -5] \, x$

b) $f'(x) = \begin{bmatrix} 0 & 2x_2 \\ \cos x_1 & -1 \end{bmatrix} \Rightarrow f'(x_0) = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$

$f(x_0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, then

$l(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + f'(x_0)(x - x_0) = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} x$

c) $f = \ln(x^T x)$; $f' = \frac{2x^T}{x^T x} \Rightarrow f'(x_0) = \frac{2}{n} x_0^T$ where $x_0 = [1 \cdots 1]^T$

and $f(x_0) = \ln(n)$ so $\ln(x) = \ln(n) + \frac{2}{n} x_0^T (x - x_0)$

$= (\ln(n) - 2) + \frac{2}{n} x_0^T x$

3) $\nabla f = \begin{bmatrix} x_2^3 - 7 & x_2 e^{x_1} \cos(x_2 e^{x_1} - 1) \\ 3x_2^2 (x_1 + 3) & e^{x_1} \cos(x_2 e^{x_1} - 1) \end{bmatrix}$; $\nabla f(x_0) = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$

$f(x_0) = \begin{bmatrix} -18 \\ 0 \end{bmatrix}$ then $p = -\nabla f(x_0)^{-1} \cdot f(x_0)$

$\Rightarrow p = \begin{bmatrix} -6/5 \\ 6/5 \end{bmatrix}$

then $x_1 = x_0 + p = \begin{bmatrix} -6/5 \\ 11/5 \end{bmatrix}$

4) $L(x_*) = F(x_k) + F'(x_k)(x^* - x_k)$

$\Rightarrow \|F(x_*) - L(x_*)\| = \|\underset{0}{\cancel{F(x_*)}} - F(x_k) - F'(x_k)(x^* - x_k)\|$

$= \|F(x_k) - F'(x_k)(x^* - x_k)\|$ (1)

we know that $\|F'(x)^{-1}\| \le M \quad \forall x \in \mathbb{R}^n$

multiply (1) by $\|F'(x)^{-1}\|$

then we get

$\|F'(x_k)^{-1}\| \, \| -F(x_k) - F'(x_k)(x^* - x)\|$ (2)

$\le M \|F(x_*) - L(x_*)\|$

Also (2) $\ge \| -F'(x_k)^{-1} F(x_k) - (x^* - x_k)\|$

(by sub multiplicative prop.)

$= \|x_{k+1} - x_k - x^* + x_k\|$

$= \|x_{k+1} - x^*\|$

So $\|x_{k+1} - x^*\| \le M \|F(x^*) - L(x^*)\|$

5) MATLAB

6) i) $F'(x) = \begin{bmatrix} 2x_1 & 4/3\, x_2^{1/3} \\ 2(x_1-1) & 2(x_2-1) \end{bmatrix} \Rightarrow F'\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

$\neq 0$

ii) Let's check $F'(x)$ is Lipschitz cont. or not

$$\|F'(y) - F'(x)\|_\infty = \left\| \begin{matrix} 2(y_1-x_1) & 4/3\,(y_2^{1/3} - x_2^{1/3}) \\ 2(y_1-x_1) & 2(y_2-x_2) \end{matrix} \right\|_\infty$$

$$= \max\left(2(y_1-x_1+y_2-x_2),\ 2(y_1-x_1) + \frac{4}{3}(y_2^{1/3} - x_2^{1/3})\right.$$

If $2(y_2-x_2) \geq \frac{4}{3}(y_2^{1/3} - x_2^{1/3})$ on some n-hood of ~~$x*$~~ - then

$$\|F'(y) - F'(x)\|_\infty = 2(y_1-x_2) + 2(y_2-x_2)$$
$$\leq 4\|y-x\|_\infty$$

If $\frac{4}{3}(y_2^{1/3} - x_2^{1/3}) \geq 2(y_2-x_2)$, on some n-hood of $x*$.

Note that $f(x) = \sqrt[3]{x}$ is lipschitz cont

i.e $\exists c \geq 0$ s.t $(y_2^{1/3} - x_2^{1/3}) \leq c(y_2-x_2)$

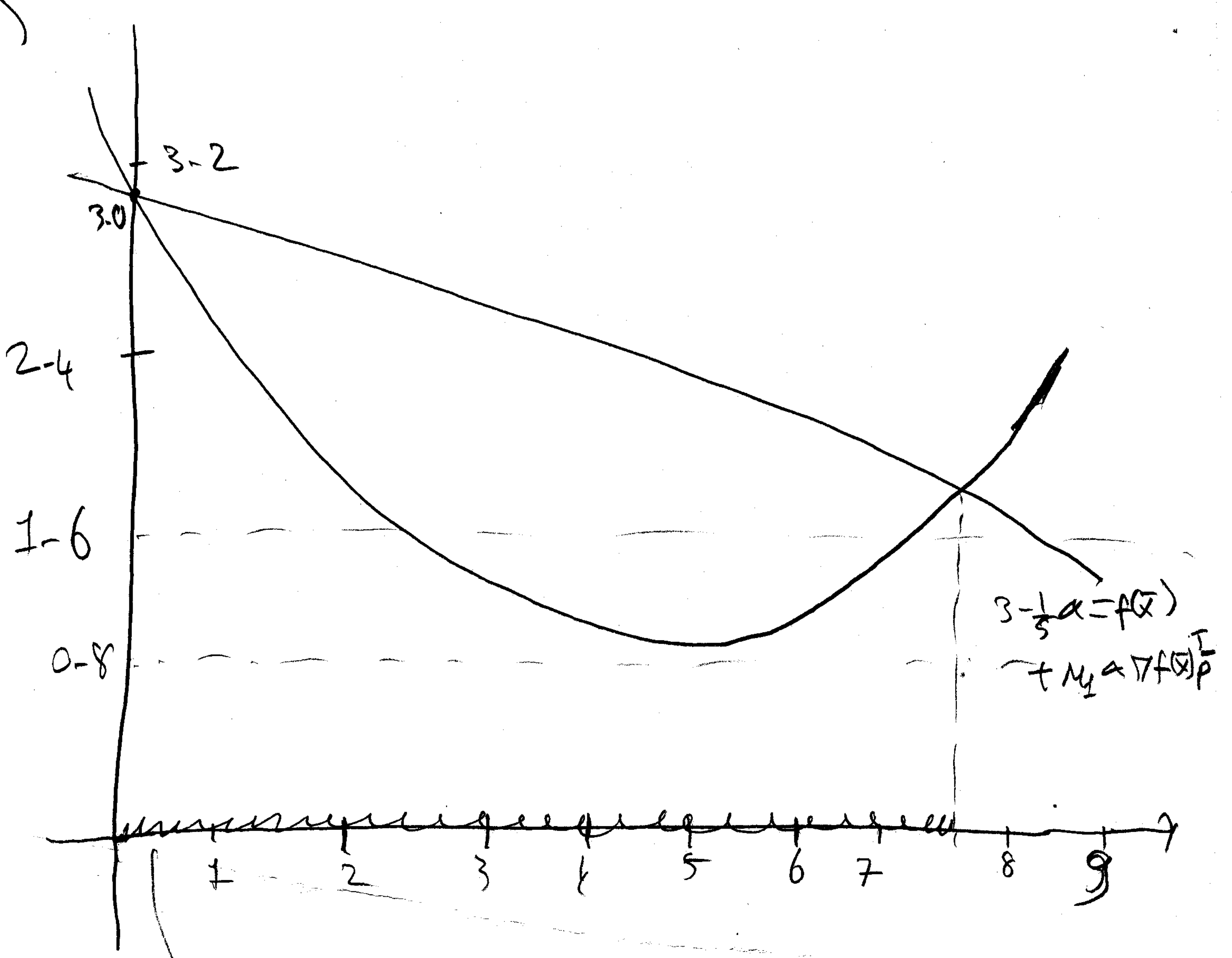
$$\Rightarrow \frac{4}{3}(y_2^{1/3} - x_2^{1/3}) \leq C_*\cdot(y_2-x_2)$$

$$\Rightarrow \|F'(y) - F'(x)\|_\infty \leq \underbrace{(C_*+2)}_{\tilde{C}}\|x_2-y_2\|$$

Therefore, $F'(x)$ is Lipschitz cts.

Hence, the convergence is quadratic.

7-)



step lengths satisfying

$$f(\bar{x}+\alpha\bar{p}) \le f(\bar{x}) + M_1 \alpha \nabla f(\bar{x})^T \bar{p}$$

8) a) Let $Q(\alpha) = f(x_k + \alpha p)$

and let $l(\alpha) = Q(0) + \alpha Q'(0)$

we know from Lecture Notes

$$\lim_{\alpha \to 0} \frac{Q(0) - Q(\alpha)}{l(0) - l(\alpha)} = 1 \qquad \text{If } M \in (0,1)$$

then $$\frac{Q(0) - Q(\alpha)}{l(0) - l(\alpha)} \geq M$$ (for small α) (by continuity of f

then $Q(0) - Q(\alpha) \geq (l(0) - l(\alpha)) M$ equivalently
$f(x_k) - f(x_k + \alpha p) \geq \alpha M \nabla f(x_k)^T p_k$ which is

Armijo sufficient decrease condition.

b) immidiately follows from a)

9) $\nabla f = \begin{bmatrix} x_2^2 e^{x_1} \\ 2x_2 e^{x_1} \end{bmatrix}$, $\nabla^2 f = \begin{bmatrix} x_2^2 e^{x_1} & 2x_2 e^{x_1} \\ 2x_2 e^{x_1} & 2e^{x_1} \end{bmatrix}$

$p_0^{SD} = -\nabla f(x_0) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, $\nabla^2 f(x_0)\, p_0^N = -\nabla f(x_0)$

$, \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} p_0^N = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \Rightarrow p_0^N = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Now; $\nabla f(x_0)^T p_0^N = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -1 < 0 \Rightarrow p_0^N$ is a descent direction.

the quadratic model used by Newton's Method:
$$Q(x) = f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0)$$

assume x_* is minimizer of $Q(x)$ then by second order necessary cond.
$\nabla Q(x_*) = 0$ and $\nabla^2 Q(x_*) \succeq 0$ since $\nabla^2 Q(x) = \nabla^2 f(x_0) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

$\mp\sqrt{13} + 3$ eigenvalues not all of them are pos. so $\nabla^2 Q(x_*) \not\succeq 0$

10.) $\nabla q(x_0) = A$, $\nabla q(x_0) = Ax_0 + b$, then

$$A p_0 = -A x_0 - b \Rightarrow p_0 = -x_0 - A^{-1} b \Big\}$$ A^{-1} exists since $A \succ 0$

then $x_1 = x_0 + p_0 = -A^{-1} b$

Claim: $-A^{-1}b$ is unique loc. min.

By second order sufficient conditions:

$$\nabla q \,(-A^{-1}b) = A(-A^{-1}b) + b = 0$$ (Clearly $\nabla q(x_0) = 0$ $x^* = -A^{-1}b$ is only stationary point.)(*)

$$\nabla^2 q \,(-A^{-1}b) = A \succ 0$$

then $-A^{-1}b$ is a local minimizer

by (*) $-A^{-1}b$ is only stationary point then

$-A^{-1}b$ is the unique local minimizer.

11) MATLAB

12) MATLAB